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Convergence and Descent in the Fermat Location Problem

LAWRENCE M. OSTRESH, Jr.

The University of Wyoming, Laramie

The Fermat location problem is to find a point whose sum of weighted distances from m given points (vertices) is a minimum. The best known method of solution is an iterative scheme devised by Weiszfeld in 1937, which converges to the unique minimum point unless one of the iterates happens to "land" on a nonoptimal vertex. The convergence proof of this scheme depends on two theorems, one of which (descent theorem) states that the objective function strictly decreases at each step. This paper extends the descent theorem by proving: (1) there is a "ball" whose radius and center depend on the Weiszfeld iteration, such that any algorithm whose iterates are "in the ball" or "on its surface" is a descent algorithm; (2) under certain circumstances, one or more of the vertices may be deleted, although the weight(s) are taken into account, and the Weiszfeld algorithm retains its descent property. In general there are several subsets of vertices which may be deleted, and for each subset, a corresponding iterate; (3) the convex hull of these iterates is such that all points within it have the descent property. Examples of the potential application of these extensions are given, including the construction of a modified Weiszfeld algorithm that without exception converges to the optimum. Beyond that, it is hoped the theorems may in time be useful in proving the descent property of yet to be discovered, very fast, nongradient algorithms.

The Fermat location problem is to find a point whose sum of weighted distances from m given points is a minimum. The problem is very old, the case $m = 3$ having been posed by Fermat and solved by Torricelli prior to 1640! (See ZACHARIAS[19] for an account of the early mathematical literature.) While the problem is nonlinear, it has a dual, first stated by Fasbender in 1846, which according to KUHN[7] has "... almost all of the useful properties of the duality of linear programming (p. 48)." And it has no known closed-form solution—in particular, the center of gravity of the weights and points is *not* ordinarily a solution.

The problem and means of solving it have attracted much interest of late, especially in the fields of spatial economics and geography, where it is known as the generalized Weber problem (after ALFRED WEBER,[17] the founder of industrial location theory). Potential applications include the siting of steel mills so as to minimize transport cost, and state capitals so as to maximize accessibility. Further, it has been extended in numerous ways: COOPER[3] has considered the minimization of weighted sums of positive powers of distance; PLANCHARD AND HURTER[13] dealt with mixed norm problems; and KATZ[5] proved the local convergence of a generalized algorithm for minimizing analogous sums involving any one of a wide class of increasing functions of distance. See LEA[10] for a recent annotated bibliography of the problem, its applications, and its extensions. For reviews, see Refs. [14] and [15].

The best known method of solution is an iterative gradient scheme devised by WEISZFELD[18] in 1937. While Weiszfeld proved the convergence of his algorithm, his work remained largely unknown in the United States until the early sixties—by which time the algorithm had been independently rediscovered, although not proven to converge, by at least MIEHLE,[11] KUHN AND KUENNE,[9] and COOPER.[2]

Weiszfeld's convergence proof contains an error, first noted in Ref. [9], and subsequently corrected by KUHN.[8] The corrected proof shows that the algorithm converges to the unique minimum point, *unless* it happens to "land" on a nonoptimal given point. The proof of convergence depends on two key theorems: (1) that the objective function strictly decreases at each step; and (2) that the sequence of iterates does not *converge* to a nonoptimal member of the set of m given points (although it may "land and get stuck" on such a point).

Elsewhere [Ref. 12] I have shown how to avoid the vertex iterate problem in the Weiszfeld algorithm. There it was also proved that the step-lengths of the algorithm may be as much as doubled, while still retaining their descent property. In the present paper these earlier results appear as special cases of far more general descent theorems. It is hoped that these theorems may be useful in proving the descent property for some still to be discovered very fast, nongradient iterative method. That such methods may soon be forthcoming is suggested by the work of HARRIS.[4]

1. FORMALIZATION

LET $A_i = (a_{1i}, \cdots, a_{pi})$, $i = 1, \cdots, m$, be a given set of distinct points called *vertices* in p-dimensional Euclidean space E^p; and let $w_1, \cdots, w_m$ be an associated set of positive scalars called *weights*. The Fermat location problem then is to find $P = (x_1, \cdots, x_p)$ such that

$$f(P) = \textstyle\sum_i w_i \overline{PA}_i$$

is a minimum; where $\overline{PA}_i = \| P - A_i \| = [(x_1 - a_{1i})^2 + \cdots + (x_p - a_{pi})^2]^{1/2}$ is the Euclidean distance between P and A_i. Call any point which minimizes f a *minimum point* and designate it M.

It follows from elementary properties of norms that f is convex. If the set of vertices is noncollinear, f is *strictly* convex and in this case there is a unique minimum point. We consider only this case.

THEOREM 1. *$P = M$ iff*

(a) *P is not a vertex and $G(P) = 0$; where*

$$G(P) = \textstyle\sum_i w_i(P - A_i)/\overline{PA}_i; \text{ or}$$

(b) *P is a vertex, say A_j, and $\| G_j(A_j) \| \le w_j$, where*

$$G_j(A_j) = \textstyle\sum_{i \neq j} w_i(A_j - A_i)/\overline{A_jA_i}.$$

For a recent proof of Theorem 1, see Kuhn[8]; for a much older proof, see STURM.[16]

2. WEISZFELD ALGORITHM AND PROOF

NOTE that $G(P)$ is the gradient of f evaluated at P, and that the requirement $G(M) = 0$, M not a vertex, is merely the first-order minimization condition on f. Setting $G(P) = 0$ and "solving for P" leads directly to the Weiszfeld transformation $W: E^p \to E^p$, defined by

$$\begin{aligned} W(P) &\equiv \frac{\sum_i w_i A_i/\overline{PA}_i}{\sum_i w_i/\overline{PA}_i}, \; P \text{ not a vertex;} \\ &\equiv P - G(P)/(\textstyle\sum_i w_i/\overline{PA}_i). \end{aligned} \tag{1}$$

If P is a vertex, say A_k, then continuity requires setting $W(A_k) = A_k$; for a proof, multiply both numerator and denominator in (1) by PA_k and then let $P \to A_k$. The associated Weiszfeld iteration is then $P_{r+1} = W(P_r)$, $r = 0, 1, \ldots$. It is apparent that the algorithm "sticks" at a non-optimal vertex if it lands there.

The theorem to be extended states that f strictly decreases at each iteration. More precisely

THEOREM 2 (*Descent Theorem*). *Let $P' \equiv W(P)$. Then $P' \neq P$ implies $f(P') < f(P)$.*

Proof (*Weiszfeld*). If P is not a vertex, then from (1),

$$P' = \frac{\sum_i u_i A_i}{\sum_i u_i};$$

where $u_i = w_i/\overline{PA_i}$, $i = 1, \cdots, m$. P' is thus the center of gravity of a system of points with positive weights and therefore uniquely minimizes the weighted sum of *squares* of distances. Hence

$$\textstyle\sum_i u_i \overline{P'A_i}^2 \leq \sum_i u_i \overline{PA_i}^2;$$
$$\textstyle\sum_i w_i \overline{P'A_i}^2/\overline{PA_i} \leq f(P);$$

the equality holding iff $P' \equiv P$.

From the identity

$$\overline{P'A_i}^2 \equiv [(\overline{P'A_i} - \overline{PA_i}) + \overline{PA_i}]^2, \tag{2}$$

we obtain $\sum_i w_i(\overline{P'A_i} - \overline{PA_i})^2/\overline{PA_i} + 2\sum_i w_i(\overline{P'A_i} - \overline{PA_i})\overline{PA_i}/\overline{PA_i} + \sum_i w_i \overline{PA_i}^2/\overline{PA_i} \leq f(P)$; $\frac{1}{2}\sum_i w_i(\overline{P'A_i} - \overline{PA_i})^2/\overline{PA_i} + f(P') \leq f(P)$. Equality holds iff $P' \equiv P$. The left-hand summation is non-negative, and the theorem follows.

3. EXTENSIONS OF THE DESCENT THEOREM

LET T be *any* transformation T: $E^p \to E^p$ and denote $P' \equiv T(P)$. If in particular T is the Weiszfeld transformation (1), then let $C \equiv W(P)$. Furthermore, let $A \cdot B$ denote the dot product of any two vectors A and B, and let $A^2 \equiv A \cdot A$. Finally, let E^n denote the minimum dimensional subspace of E^p containing the vertices, $E^n \subseteq E^p$, $2 \leq n \leq p$, $n \leq m - 1$.

THEOREM 3. *If T is such that $\overline{CP'} \leq \overline{CP}$; if P', $P \in E^n$; if P is not a vertex; and if $P' \neq P$; then*

$$f(P') < f(P).$$

Proof. By direct calculation $\overline{CP'} \leq \overline{CP} \to (C - P')^2 \leq (C - P)^2 \to C^2 - 2C \cdot P' + P'^2 - C^2 + 2C \cdot P - P^2 \leq 0 \to P'^2 - P^2 - 2(P' - P) \cdot C \leq 0$, and so from (1)

$$\textstyle\sum_i w_i[(P'^2 - 2P' \cdot A_i + A_i^2) - (P^2 - 2P \cdot A_i + A_i^2)]/\overline{PA_i} \leq 0;$$

or

$$\textstyle\sum_i w_i \overline{P'A_i}^2/\overline{PA_i} \leq \sum_i w_i \overline{PA_i} = f(P).$$

The equality holds only if $\overline{CP'} = \overline{CP}$. Use (2) as in the proof of Theorem 2 to obtain

$$\textstyle\frac{1}{2}\sum_i w_i(\overline{P'A_i} - \overline{PA_i})^2/\overline{PA_i} + f(P') \leq f(P).$$

The summation is non-negative. Moreover, since P', $P \in E^n$ and $P' \neq P$, then by Lemma 1 (below) there exists at least one vertex, say A_k, such that $\overline{P'A_k} \neq \overline{PA_k}$; hence the summation is strictly positive and the theorem follows.

LEMMA 1. *If P_1, $P_2 \in E^n$ then $\overline{P_1A_i} = \overline{P_2A_i}$, $i = 1, \cdots, m$ iff $P_1 = P_2$.*

Proof. Let B be any set of $n + 1$ linearly independent vertices which determine E^n. Borsuk[1] (p. 127) proves that the position of each point $P \in E^n$ is uniquely determined by the $n + 1$ distances $\overline{PA_j}$, $A_j \in B$. Hence, the position of each point $P \in E^n$ is uniquely determined by the $m \geq n + 1$ distances $\overline{PA_1}, \cdots, \overline{PA_m}$.

Theorem 3 admits of a simple geometric interpretation: there is a "ball" of radius $\overline{CP}$, centered at C; if $P' = T(P)$ is "in the ball" or "on its surface" (but $P' \neq P$) then $f(P') < f(P)$. Furthermore, when one recalls that C is the center of gravity of a system of points with positive weights, it should be intuitively clear why Theorem 3 is true, if Theorem 2 is: the graph of $h(P') = \sum_i u_i \overline{P'A_i}^2$, $u_i \equiv w_i\overline{PA_i}$ is an "upward" facing paraboloid of revolution centered at C; points closer to C than is P are on lower level curves of h than is P. The situation for points in E^2 is portrayed in Figure 1.

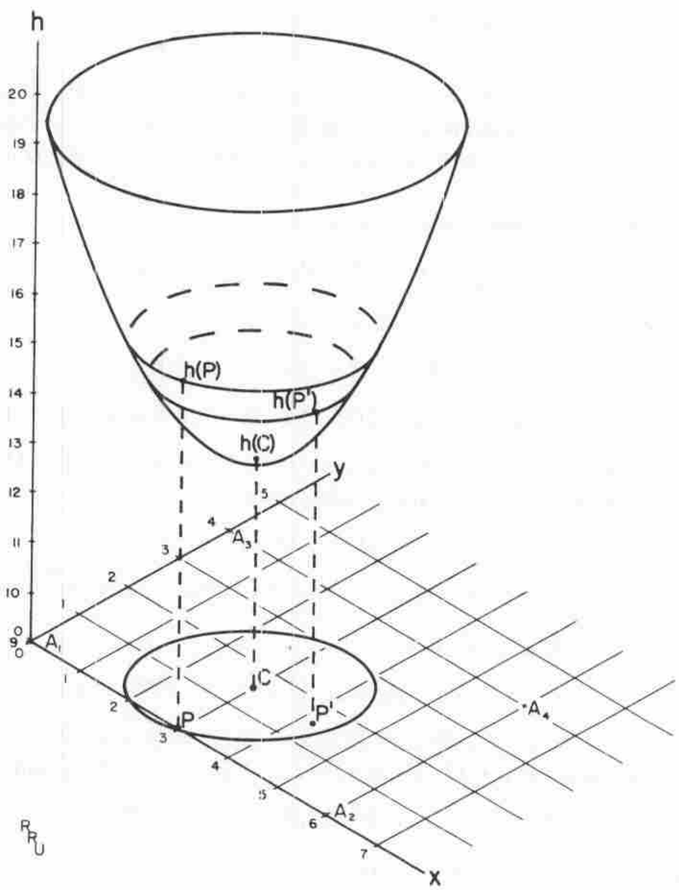

Fig. 1. Paraboloid of revolution generated by $h(P')$.

One may question the need for the condition $P, P' \in E^n$ and for Lemma 1 because it may seem obvious that $\sum_i w_i(\overline{P'A_i} - \overline{PA_i})^2/PA_i \neq 0$ if $P' \neq P$, P not a vertex. In fact, it is neither obvious nor in general true! As a counter-example suppose $A_1 = (0,0)$, $A_2 = (1,0)$, $w_1 = w_2 = 1$, $P = (0,1)$, and $P' = (0, -1)$. Then $(1 - 1)^2/1 + (\sqrt{2} - \sqrt{2})^2/1 = 0$, but $P' \neq P$.

Further results are aided by the following notation: Let I be the index set of the weights and vertices, let J be a nonempty subset of I, and let K be the complement of J, so that $J \cup K = I$, $J \cap K = \phi$, where ϕ is the null set. Let $G_J(P)$ denote a "subset gradient" of $f(P)$, defined by

$$G_J(P) = \textstyle\sum_{j\in J} w_j(P - A_j)/\overline{PA_j},\ (P \neq A_j,\ \forall\, j \in J).$$

Further, for K not empty and $G_J(P) \neq 0$ (we shall only be interested in such cases), let $U_J = G_J(P)/\ \|\ G_J(P)\ \|$, a unit vector in the direction of $G_J(P)$; and define the *resultant* $R_J(P)$ as:

$$R_J(P) = -\ (G_J(P) - U_J w_K),$$

where $w_K = \sum_{k\in K} w_k$.

Note that if $\|\ G_J(P)\ \| > w_K$ then the direction of $R_J(P)$ is opposite that of $G_J(P)$; its magnitude is that of $G_J(P)$ reduced by w_K. Also note that if $J = I$ then K is empty, so $R_I(P)$ is the negative *gradient* of f, if P is not a vertex.

Finally, define the positive quantity

$$s_J(P) = \textstyle\sum_{j\in J} w_j/\overline{PA_j},\ (P \neq A_j,\ \forall\, j \in J);$$

and let $T_J(P)$ be a transformation defined by

$$T_J(P) \equiv P'_J \equiv P + 2R_J(P)/s_J(P). \tag{3}$$

LEMMA 2. *If $P \neq A_j$, $\forall\, j \in J$ and $\|G_J(P)\| \geq w_K$, then $f(P'_J) \leq f(P)$.*

Proof. If $\|G_J(P)\| = w_K$, then $R_J(P) = 0$, $P'_J = P$, and the lemma is trivially true. Otherwise, by direct calculation:

$$(P'_J - P)\ s_J\ (P) - 2R_J(P) = 0;$$

$$(P'_J - P)\ [\{\textstyle\sum_{j\in J} w_j(P'_J - P + 2P - 2A_j)/\overline{PA_j}\} - 2U_J w_K] = 0;$$

$$\{\textstyle\sum_{j\in J} w_j[(P'^2_J - 2P'_J\cdot A_j + A_j^2) - (P^2 - 2P\cdot A_j + A_j^2)]/\overline{PA_j}\} - 2(P'_J - P)\cdot U_J\, w_K = 0.$$

Since $\|\ G_J(P)\ \| > w_K$, $G_J(P)$ and U_J have directions opposite $R_J(P)$ and therefore opposite $(P'_J - P)$. Hence, $-(P'_J - P)\cdot U_J = \overline{PP'_J}$ and

$$\overline{2PP'_J}w_K + \textstyle\sum_{j\in J} w_j\ \overline{P'_JA_j}^2/\overline{PA_j} = \sum_{j\in J} w_j\overline{PA_j}.$$

Use (2) as in the proof of Theorem 2 to obtain

$\overline{PP'}_J \sum_{k\in K} w_k + ½ \sum_{j\in J} w_j(\overline{P'_JA_j} - \overline{PA_j})^2/\overline{PA_j} + \sum_{j\in J} w_j\overline{P'_JA_j} = \sum_{j\in J} w_j\overline{PA_j}.$

Adding $\sum_{k\in K} w_k \overline{PA_k}$ to both sides and using the triangle inequality $w_k\overline{PP'_J} + w_k\overline{PA_k} \geq w_k\overline{P'_JA_k}, k\in K$, gives $½ \sum_{j\in J} w_j (\overline{P'_JA_j} - \overline{PA_j})^2/\overline{PA_j} + f(P'_J) \leq f(P)$,

which, since the summation is non-negative, proves the lemma.

We show below that if P is a vertex there exists J such that the conditions of Lemma 2 apply. Here we more generally prove:

LEMMA 3. *If P is an interior point of the convex hull of the vertices, and if P is not a vertex, then there exists $J\subset I$ such that* $\| G_J(P) \| \geq w_K$.

Proof. Let K consist of the single element k, and thus $J = I - \{k\}$. From the relation $G(P) = G_J(P) + w_k (P - A_k)/\overline{PA_k}$ we obtain

$$\| G(P) \|^2 - 2w_k \| G(P) \| \cos\alpha_k + w_k^2 = \| G_J(P) \|^2,$$

where α_k is the angle between $G(P)$ and $(P - A_k)$. If $P = M$, then $\| G(P) \| = 0$ and $w_k = \| G_J(P) \|$. Otherwise, since P is an interior point, there exists at least one vertex, say $A_{\hat{k}}$, such that $\cos\alpha_{\hat{k}} \leq 0$, which, since $\| G(P) \| > 0$, proves the lemma.

In general there are several subsets $J \subset I$ for which the conditions of Lemma 2 hold, and for each such subset there is a corresponding point P'_J. Denote the set of all such points by $P^* \equiv \{P'_J\}$ (formally, $P'_J \in P^*$ iff $J \subset I$ is such that $\| G_J(P) \| \geq w_K, K \equiv I - J$). Further, let H denote the convex hull of the points in $P^* \cup \{P\}$. Then by the (strict) convexity of f, we have

THEOREM 4. *$Q \in H$ and Q not an extreme point of H implies $f(Q) < f(P)$.*

One of the difficulties with Weiszfeld's algorithm is that if an iterate lands on a non-optimal vertex it stays there. While the algorithm cannot *converge* to such a vertex, it can "jump" there—Kuhn's [8] correction to Weiszfeld's convergence proof consists exactly of pointing this out and restricting the generality of the proof accordingly. As a specific application of Theorem 4, observe how it solves this problem of vertex iterates:

First note that if K is empty, (3) reduces to $P'_I = P - 2G(P)/s_I(P)$, so that $P' = (P + P'_I)/2$. Comparison with (1) suggests the following generalization of the Weiszfeld transformation:

$$C_J \equiv W_J(P) \equiv (P + P'_J)/2 = P + R_J(P)/s_J(P). \qquad (4)$$

Suppose that $P = A_k \neq M$: then by Theorem 1, $\| G_k(A_k) \| > w_k$. Let K consist of the single element k; since $J = I - K$ then $A_k \neq A_j, \forall j \in J$, and

$G_k(A_k) = G_J(A_k)$. The conditions of Lemma 2 apply and $P'_J \in P^*$; thus $C_J \in H$, since $C_J = (A_k + P'_J)/2$. Also, $R_J(A_k) \neq 0$ and $s_J(A_k)$ is bounded, so $C_J \neq A_k$ and $C_J \neq P'_J$. Therefore, C_J is not an extreme point of H and by Theorem 4 $f(C_J) < f(A_k)$.

The Weiszfeld algorithm is known to converge to M provided there are no vertex iterates. The use of (4) in the event of such an iterate would provide an algorithm that *without exception* converges to the optimum.

4. EXAMPLES

IN ORDER to fix ideas and to show the potential applicability of the preceding results, we present two examples. In both cases we assume coplanar vertices and for notational convenience set $A_i = (a_i, b_i)$ and $P = (x, y)$.

The first example demonstrates the construction of H, the convex hull of points generated by (3). Let the vertices and weights be given as in Table I, and assume $P = (1,0)$. Suppose $J = \{1,3,4,6\}$; then $K = \{2,5\}$, and from Table I:

$$G_J(P) = \begin{bmatrix} 3 + 0.7071 + 0.7071 - 0.7071 \\ 0 - 0.7071 + 0.7071 + 0.7071 \end{bmatrix} = \begin{bmatrix} 3.7071 \\ 0.7071 \end{bmatrix};$$
$$\| G_J(P) \| = 3.7739 > 2 = \textstyle\sum_{k\in K} w_k.$$

Since $P'_J \in P^*$, we proceed to calculate P'_J:

$$s_J(P) = 1.5 + 0.7071 + 0.7071 + 0.7071 = 3.6213;$$
$$R_J(P) = -(G_J(P) - U_J \textstyle\sum_{k\in K} w_k) = -G_J(P)(1 - (\sum_{k\in K} w_k)/ \| G_J(P) \|);$$
$$R_J(P) = -\begin{bmatrix} 3.7071 \\ 0.7071 \end{bmatrix} (1 - 2/3.7739) = \begin{bmatrix} -1.7425 \\ 0.3324 \end{bmatrix}.$$

By (3), then:

$$P'_J = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{2}{3.6213} \begin{bmatrix} -1.7425 \\ -0.3324 \end{bmatrix} = \begin{bmatrix} 0.0376 \\ -0.1836 \end{bmatrix}.$$

TABLE I

DATA FOR CONSTRUCTION OF H

$P \equiv (x,y) = (1,0) \qquad A_i \equiv (a_i, b_i)$

i	Vertices		Weights	$w_i/\overline{PA_i}$	Gradient	
	a_i	b_i			$w_i(x - a_i)/PA_i$	$w_i(y - b_i)/\overline{PA_i}$
1	−1	0	3	1.5	3.0	0
2	1.1	0	1	10.0	−1.0	0
3	0	1	1	0.7071	0.7071	−0.7071
4	0	−1	1	0.7071	0.7071	0.7071
5	2	1	1	0.7071	−0.7071	−0.7071
6	2	−1	1	0.7071	−0.7071	0.7071

Now suppose that $J = \{2,3,4,5,6\}$; then $K = \{1\}$, $G_J(P) = (-1,0)$, and $\| G_J(P) \| = 1 < 3 = \sum_{k \in K} w_k$. In this case, $P'_J \notin P^*$ so we stop at this point.

All other P'_J are constructed in a like manner. By enumerating all possible subsets $J \subset I$, the elements of P^* are determined. Table II gives a full listing of all $P'_J \in P^*$ for this example. Figure 2 plots them and the associated convex hull H. In this case $M = (-0.0978, 0.0)$ and appears as a star in the figure.

For the second example, we consider the vertex iterate problem referred to earlier. Let $A_1 = (-2,0)$, $A_2 = (-1,0)$, $A_3 = (1,0)$, $A_4 = (2,0)$, $A_5 = (0, 1)$, $A_6 = (0,-1)$, and let $w_1 = \cdots = w_6 = 1$. These data, which first appeared in Ref. [8], are plotted in Figure 3. By symmetry, $M = (0,0)$. Also, if $P = (x,0)$, then the Weiszfeld transformation is such that $W(P) = P' = (x',0)$; that is, the sequence of iterates is one-dimensional.

Figure 4 (based on Figure 2 in Ref. [8]) plots x' as a function of x (dashed line), for P between M and A_4. Notice that for some P, approximately (1.62,0), $W(P) = A_3$, and thus the algorithm is "stuck."

The generalized Weiszfeld transformation suffers from no such difficulty, however. If $P = A_k$, a test can be made to determine whether $\| G_k(A_k) \| \le w_k$: if it is, then $A_k = M$, by Theorem 1; if it is not, then set $K = \{k\}$ and use $W_J(A_k)$, as given in (4). In this example,

TABLE II

ELEMENTS OF P^*

J	K	$G_J(P)$		$\sum_{k \in K} w_k$	s_J	P'_J	
		$\partial f/\partial x$	$\partial f/\partial y$			x'_J	y'_J
1,2,3,4,5,6	ϕ	2	0	0	14.328	0.721	0
1,3,4,5,6	2	3	0	1	4.328	0.076	0
1,2,4,5,6	3	1.293	0.707	1	13.621	0.939	−0.033
1,2,3,5,6	4	1.293	−0.707	1	13.621	0.939	0.033
1,2,3,4,6	5	2.707	0.707	1	13.621	0.745	−0.067
1,2,3,4,5	6	2.707	−0.707	1	13.621	0.745	0.067
1,4,5,6	2, 3	2.293	0.707	2	3.621	0.789	−0.065
1,3,5,6	2, 4	2.293	−0.707	2	3.621	0.789	0.065
1,3,4,6	2, 5	3.707	0.707	2	3.621	0.038	−0.184
1,3,4,5	2, 6	3.707	−0.707	2	3.621	0.038	0.184
1,2,4,6	3, 5	2	1.414	2	12.914	0.943	−0.040
1,2,4,5	3, 6	2	0	2	12.914	1	0
1,2,3,6	4, 5	2	0	2	12.914	1	0
1,2,3,5	4, 6	2	1.414	2	12.914	0.943	0.040
1,2,3,4	5, 6	3.414	0	2	12.914	0.781	0
1,4,6	2,3,5	3	1.414	3	2.914	0.803	−0.093
1,4,5	2,3,6	3	0	3	2.914	1	0
1,3,6	2,4,5	3	0	3	2.914	1	0
1,3,5	2,4,6	3	−1.404	3	2.914	0.803	0.093
1,3,4	2,5,6	4.414	0	3	2.914	0.030	0

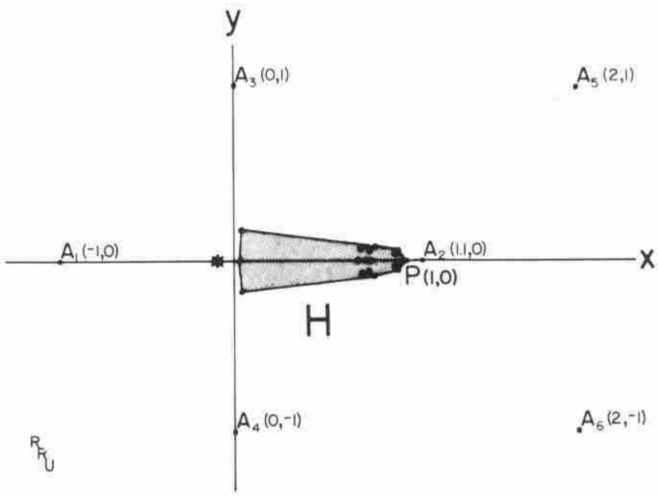

Fig. 2. Construction of convex hull H.

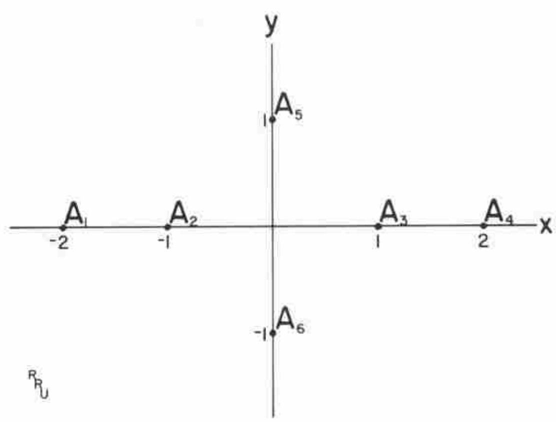

Fig. 3. Location of vertices.

$$C_J = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{4.66} \begin{bmatrix} -2.83 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.393 \\ 0 \end{bmatrix}.$$

The result is plotted as a star (*) in Figure 4. Note that in this figure the closer x' is to the x-axis the "better" the algorithm, for the faster its convergence to M. We thus have the paradoxical result that vertex iterates are substantially "better" than their near neighbors. This suggests the following adaptation of Weiszfeld's algorithm: At each iteration, determine the vertex nearest to P_r. Suppose this is A_k. Then set $K = \{k\}$ and check whether $\| G_k(P) \| > w_k$. If it is, then set $P_{r+1} = W_J(P_r)$; the

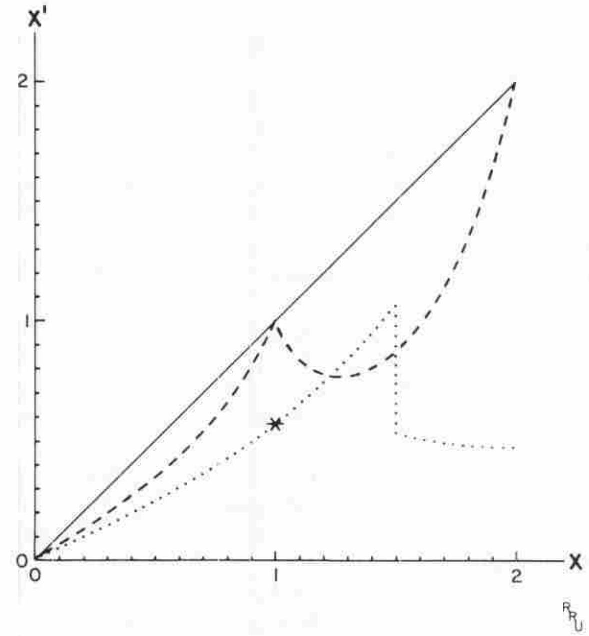

Fig. 4. x' as a function of x.

effect of this procedure, applied to the preceding example, is depicted in Figure 4 (dotted line).

It is presently not known whether this procedure (which incidentally is in general a non-gradient method) substantially improves the rate of convergence of the Weiszfeld algorithm. For a good discussion of this rate of convergence and a means of improving it, see KATZ.[6]

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