May's Theorem with an Infinite Population

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Abstract

In this paper, we investigate majority rule with an infinite number of voters. We use an axiomatic approach and attempt to extend May's Theorem characterizing majority rule to an infinite population. The analysis hinges on correctly generalizing the anonymity condition and we consider three different versions. We settle on bounded anonymity as the appropriate form for this condition and are able to use the notion of asymptotic density to measure the size of almost all sets of voters. With this technique, we define density q-rules and show that these rules are characterized by neutrality, monotonicity, and bounded anonymity on almost all sets. Although we are unable to provide a complete characterization applying to all possible sets of voters, we construct an example showing that our result is the best possible. Finally, we show that strengthening monotonicity to density positive responsiveness characterizes density majority rule on almost all sets.

1 Introduction

An important contribution to the development of social choice theory was Kenneth May's axiomatization of majority rule (May, 1952). May's Theorem characterizes majority rule as the unique rule that satisfies the conditions of anonymity, neutrality, and positive responsiveness with a finite number of voters. More generally, the class of quota rules (also known as supermajority rules) are characterized by anonymity, neutrality, and monotonicity (Fishburn, 1974). In this paper, we investigate majority rule with a countably infinite population by reconsidering the axioms of May's Theorem in this context.

The focus of this paper is majority rule with a countably infinite population. Specifically, we assume throughout that the set of voters is $\mathbb{N}$, the set of natural numbers. Some papers explicitly work with this set of voters (Callander, 2002), while other interpretations of such an infinite society include future generations (Koopmans, 1960; Diamond, 1965; Svensson, 1980), finitely many long-lived organizations such as political parties or firms, or finitely many voters with uncertainty given by an infinite number of states of the world (Mihara, 1997b). For example, interest in intergenerational equity may lead one to argue that a societal change should be implemented only if it makes a majority of future generations better off. Given our through understanding of majority rule in finite groups, it may be surprising that there is much to say about the extension to infinite societies. However, this question turns out to be quite subtle in several different ways.

The first unexpected problem that we must face is that it is difficult to correctly define "majority rule" in an infinite setting, even with only two alternatives. With a finite number of voters, majority rule over two alternatives is easily specified as choosing the alternative with the most votes. Put another way, the majority procedure reduces to selecting the alternative for which the set of voters supporting it is larger. With a finite number of voters, this decision involves nothing more than counting the number of voters supporting each alternative and picking the larger of the two. However, with an infinite number of voters, things are not nearly so simple. As first recognized by Cantor, every countably infinite set is the same "size".[1] For example, the set of even integers has the same size as the set of all integers. This may seem surprising, since the latter set has all the elements of the former set plus all the odd integers as well, but this is just a indication that the realms of finite sets and infinite sets can be very different.

So, for majority rule over two alternatives, in the finite case the question is simply which alternatives gets more votes. In the realm of infinite sets, however, "more" is a tricky concept: one can add elements to an infinite set without altering its size. In this way, moving from a finite setting to an infinite one turns the simple question of defining majority rule into a quite perplexing one. For example, let $T = \{3, 6, 9, \dots\}$ denote the set of every third integer and $R = \{1, 2, 4, 5, 7, 8, \dots\}$ denote the remaining numbers. Intuitively, the latter set forms a majority of the natural numbers and, thus, if this set represented the voters in favor of one alternative, we would expect that alternative to be the winner by majority rule. However, the standard theory of infinite sets holds that T and R have equal size and that neither set has "more" elements than the other. Thus, the definition of majority rule as the alternative that gets more votes that is so obviously appropriate in the finite case is unsuitable for infinite societies.

Setting aside this definitional question, a second problem that must be dealt with emerges from considerations of anonymity. Loosely speaking, majority rule should treat all voters equally; only the number of votes an alternative receives should matter and not the identity of those casting the votes. Introduced by May, the standard anonymity axiom requires that the outcome not change if we interchange the votes of some or all of the voters. Equiva-

[1]Here we are using the standard set theoretic notion of cardinality as our conception of "size." In particular, two sets have the same cardinality precisely when there is a bijection between them. For all finite sets this gives us the usual definition of "size" as the number of elements in the set.

lently, it requires that the outcome be invariant under rearrangements of the set of voters. While this requirement is straightforward with a finite population, once again, in the infinite setting it becomes problematic. Suppose we consider the ballots cast by the voters as an infinite sequence of 0's and 1's, where the kth element of the sequence is the vote cast by voter k. Then it is a simple exercise to show that any such sequence containing an infinite number of both 0's and 1's can be rearranged into any other such sequence. But then the standard anonymity axiom would require any such situation be assigned the same outcome. Referring back to the sets T and R defined above, the standard anonymity axiom, combined with neutrality, would require that both of these sets be assigned a tie, rather than the commonsense choice of R as a majority. Even if only every millionth voter was in favor of an alternative, the standard version of these two axioms would still require a tied outcome. Clearly, the standard anonymity axiom is too strong. Because of this, we consider two modifications of the standard anonymity axiom, namely finite anonymity and bounded anonymity.

Although these two issues certainly give us pause, we still are motivated by our intuition about how majority rule should operate on certain profiles of votes. As the examples just mentioned clearly illustrate, the major task before us is to reconcile this intuition with the problems posed by the theory of infinite sets. In the end, we are able to resolve this discrepancy for almost all sets by using the concept of asymptotic density, a tool devised by number theorists as an alternative way to evaluate the size of infinite subsets of $\mathbb{N}$.

By now it should be clear that this paper touches on several different literatures in social choice theory and mathematics. In addition to the seminal work of May (1952, 1953), several scholars have offered axiomizations of majority rule and related procedures (Murakami, 1966; Maskin, 1995; Ching, 1996; Campbell and Kelly, 2000). However, none of these papers considers an infinite set of voters. Versions of Arrow's Theorem with countably infinite societies have been known for some time (Fishburn, 1970; Kirman and

Sondermann, 1972; Hansson, 1976), but these do not bear directly on the question at hand.[2]

The techniques used in this paper are most closely related to the work of Lauwers on social choice with infinite populations.[3] His work on both intertemporal choice (Lauwers, 1995, 1998a) and Arrow's Theorem (Lauwers and Van Liedekerke, 1995; Lauwers, 1997) deals directly with the same questions that arise in the present paper. In particular, Lauwers proposes the notion of bounded anonymity; a concept that we employ extensively as well.[4] In considering three versions of anonymity for infinite societies, he reaches the same conclusion that we do: bounded anonymity is the appropriate conceptualization whereas finite and strong anonymity are inappropriate.[5]

Notation, definitions, and some mathematical concepts we use are covered in the next two sections. Our analysis hinges on correctly generalizing May's anonymity axiom. Thus, section 4 is devoted to analyzing three distinct versions of this axiom. In the first two parts of this section, we give results that illustrate the undesirability of strong and finite anonymity, and show that bounded anonymity is the most satisfactory condition of the three. In the third part, we link bounded anonymity and asymptotic density in order to provide a suitable description of majority rule in infinite populations. Specifically, we define density q-rules and show that these rules are characterized by neutrality, monotonicity, and bounded anonymity on the collection of sets with density. Although we are unable to provide a complete characterization applying to all possible sets of voters, we show that our result applies to almost all such sets and construct an example showing that it is the best possible. In addition, we show that strengthening monotonicity to density

[2]See Armstrong (1980, 1985) and Brunner and Mihara (2000) for further work on Arrow's Theorem in infinite settings.

[3]Lauwers (1998b) is a useful survey of this research.

[4]In the mathematical literature, this concept is known as the Lévy group of permutations (Lévy, 1951; Obata, 1988; Blümlinger, 1996).

[5]Campbell and Fishburn (1980) and Mihara (1997a) consider anonymity and neutrality in other contexts.

positive responsiveness characterizes density majority rule on almost all sets. The final section contains concluding remarks.

2 Notation and Definitions

Consider a countably infinite society represented by $\mathbb{N} = \{1, 2, \dots\}$, the set of natural numbers. Denote the power set (the set of all subsets) of $\mathbb{N}$ by $P(\mathbb{N})$. For $A \in P(\mathbb{N})$, let $|A|$ denote the *cardinality* of A. Recall that $|A| = |B|$ if and only if there is a one-to-one mapping from A onto B.[6] Also note that if A and B are infinite subsets of $\mathbb{N}$, then $|A| = |B| = |\mathbb{N}| = \infty$.[7] Denote the set of nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For a set $A \in P(\mathbb{N})$, we denote the complement of A by A^c. If A^c is a finite set, then we say A is *cofinite*. Let $\mathcal{I} \subset P(\mathbb{N})$ be the collection of infinite sets that have infinite complement. That is, $A \notin \mathcal{I}$ if and only if A is finite or cofinite.

There are two *alternatives* (or candidates), labelled 0 and 1, with $X = \{0, 1\}$. Under the assumption that the preferences of the voters are linear orders, a *preference profile* for the voters is given by a function $v : \mathbb{N} \to \{0, 1\}$. Alternatively, we can interpret v as the *ballot* cast by the voters, with no abstentions. In either case, we can view v as infinite binary sequence.

For our analysis, it will be useful to work with $V = v^{-1}(1)$, the set of voters who prefer alternative 1 to alternative 0. Clearly, for any preference profile v, the corresponding set V is an element of $P(\mathbb{N})$ and vice versa.[8]

There are three possible *outcomes* to consider: alternative 1 is selected, alternative 0 is selected, or the alternatives are tied. Thus, an *aggregation rule* is a function $f : P(\mathbb{N}) \to \{0, 1/2, 1\}$, with the value 1/2 interpreted as a tie. An alternative interpretation is that f embodies the social preference of

[6]For details, see Hrbáček and Jech (1984).

[7]To be precise, $|\mathbb{N}| = \aleph_0$, but as all sets we consider are at most countable, there is no chance of confusion.

[8]Thus, we consider all possible subset of $\mathbb{N}$ to be observable. In contrast, see Armstrong (1980, 1985) and Mihara (1997a) for work on restricted coalitions.

the voters, as a function of their individual preferences. Although we do not permit voters to be indifferent, we do allow social indifference.[9]

We are now ready to specify the conditions we will impose on the aggregation rule.

Definition 1 *An aggregation rule f satisfies **neutrality** if, for all $A \in P(\mathbb{N})$, $f(A^c) = 1 - f(A)$.*

In other words, a neutral aggregation rule treats the two alternatives equally; if the preferences of the individuals in society are reversed, then the social preference is reversed. Thus, the outcome of the aggregation rule can not depend on the labelling of the alternatives.

Definition 2 *An aggregation rule f satisfies **monotonicity** if, for all $A, B \in P(\mathbb{N})$, $A \subseteq B$ implies $f(A) \leq f(B)$.*

A rule is monotonic if increasing the support for an alternative does not lower the alternative in the group preference. This is a "weak" monotonicity condition in that increasing the number of votes for an alternative can leave the social preference unchanged. The next condition we define strengthens this requirement:

Definition 3 *An aggregation rule f satisfies **positive responsiveness** if, for all $A, B \in P(\mathbb{N})$, $A \subsetneq B$ and $f(A) \neq 0$ implies $f(B) = 1$.*

Positive responsiveness strengthens monotonicity by imposing "fragility of ties." That is, if a configuration of preferences generates social indifference, changing (at least) one individual's preference gives strict social preference, thus "breaking the tie." For clarity, we have defined both monotonicity and positive responsiveness with respect to alternative 1, but, in combination with the neutrality axiom, the corresponding condition applies to alternative 0.

[9] We conjecture that similar results will hold when individual indifference is permitted.

A *permutation* on $\mathbb{N}$ is a mapping $\pi : \mathbb{N} \to \mathbb{N}$ that is one-to-one and onto. Let $\mathbb{G}$ denote the group of all permutations on $\mathbb{N}$. The image of a set $A \in P(\mathbb{N})$ under a permutation π is denoted πA. That is,

$$\pi A = \{k \in \mathbb{N} : \exists j \in A \text{ such that } \pi(j) = k\}.$$

A permutation $\pi \in \mathbb{G}$ is *finite* if there is an integer N such that $n > N$ implies $\pi(n) = n$.

Suppose $A, B \in \mathcal{I}$ and let $A = \{a_1, a_2, \dots\}$, $A^c = \{a'_1, a'_2, \dots\}$, $B = \{b_1, b_2, \dots\}$ and $B^c = \{b'_1, b'_2, \dots\}$. Then the permutation π_{AB} is defined as $\pi_{AB}(a_i) = b_i$ and $\pi_{AB}(a'_i) = b'_i$.

The Lévy group $\mathcal{G}$ is a proper subgroup of $\mathbb{G}$, defined as the group of permutations $\pi \in \mathbb{G}$ for which

$$\lim_{n \to \infty} \frac{|\{k : k \leq n < \pi(k)\}|}{n} = 0.$$

A permutation $\pi \in \mathcal{G}$ has been termed *bounded* by Lauwers (1998a,b) and we adopt the same terminology here.

The axioms we have defined so far are obvious generalizations of their finite forms. However, we will consider three different versions of anonymity in infinite societies. In general, anonymity requires that the aggregation rule be invariant under various rearrangements.

Definition 4 *An aggregation rule f satisfies*

1. ***strong anonymity*** *if, for all $A \in P(\mathbb{N})$ and all $\pi \in \mathbb{G}$, $f(\pi A) = f(A)$,*
2. ***bounded anonymity*** *if, for all $A \in P(\mathbb{N})$ and all $\pi \in \mathcal{G}$, $f(\pi A) = f(A)$,*
3. ***finite anonymity*** *if, for all $A \in P(\mathbb{N})$ and all finite permutations $\pi \in \mathbb{G}$, $f(\pi A) = f(A)$.*

Thus, strong anonymity demands that the aggregation rule be invariant under all possible permutations. At the other extreme, finite anonymity requires invariance under finite permutations. In between these two, bounded anonymity specifies invariance under not just finite permutations, but also infinite permutations with a certain limited scope of rearrangement.

3 Mathematical Preliminaries

In this section, we cover some mathematical concepts that are the basis of our results. We begin with a definition of an *ultrafilter* on $\mathbb{N}$.[10]

Definition 5 *An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets that satisfies*

1. $\emptyset \notin \mathcal{U}$ *and* $\mathbb{N} \in \mathcal{U}$,
2. *if* $A, B \in \mathcal{U}$, *then* $A \cap B \in \mathcal{U}$,
3. *if* $A \subset B$ *and* $A \in \mathcal{U}$, *then* $B \in \mathcal{U}$, *and*
4. *for all* $A \in \mathbb{N}$, *either* $A \in \mathcal{U}$ *or* $A^c \in \mathcal{U}$.

There are two very different types of ultrafilters on $\mathbb{N}$, *fixed* and *free*. An ultrafilter on $\mathbb{N}$ is fixed if each of its members contains a particular $k \in \mathbb{N}$. Any ultrafilter which is not fixed is called a free ultrafilter.[11] Equivalently, an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is free if it has empty intersection; that is, if $\cap_{A \in \mathcal{U}} A = \emptyset$.

Several useful properties of ultrafilters are collected in the following lemma. These properties are well known, consequently the proofs are omitted.

Lemma 1 *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Then the following hold:*

1. *if* $A \cup B \in \mathcal{U}$, *then either* $A \in \mathcal{U}$ *or* $B \in \mathcal{U}$;

[10] See Aliprantis and Border (1994) for more on ultrafilters. In social choice theory, Kirman and Sondermann (1972), Hansson (1976), and Brown (1975) examine ultrafilters of decisive sets in the context of Arrow's Theorem with infinite populations.

[11] Some authors use the term principal for fixed and nonprincipal for free.

2. *if* $A \cup B \in \mathcal{U}$ *and* $A \cap B = \emptyset$*, then exactly one of* $A \in \mathcal{U}$ *or* $B \in \mathcal{U}$ *holds.*

3. *if* $\mathcal{U}$ *is a free ultrafilter on* $\mathbb{N}$ *and* A *is finite, then* $A \notin \mathcal{U}$*;*

One can show that free ultrafilters on $\mathbb{N}$ exist, but the proofs involve the axiom of choice in the form of Zorn's Lemma, so explicit examples of free ultrafilters cannot be given. Nonetheless, almost all ultrafilters on $\mathbb{N}$ are free.

An ultrafilter on $\mathbb{N}$ can be viewed as a particular type of measure on $\mathbb{N}$. A *finitely additive measure* on $\mathbb{N}$ is a function $\mu : P(\mathbb{N}) \to [0,1]$ such that $\mu(\emptyset) = 0$, $\mu(\mathbb{N}) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for every two disjoint subsets A and B of $\mathbb{N}$.[12] A *purely finitely additive measure* on $\mathbb{N}$ is a finitely additive measure μ such that $\mu(A) = 0$ for every finite set $A \in P(\mathbb{N})$. Finally, a finitely additive measure μ is *zero-one* if $\mu(A) \in \{0,1\}$ for all $A \in P(\mathbb{N})$.

The set of zero-one finitely additive measures on N is exactly the set of ultrafilters on $\mathbb{N}$. For an ultrafilter $\mathcal{U}$, the corresponding zero-one finitely additive measure is given by

$$\mu(A) = \begin{cases} 1 & \text{if } A \in \mathcal{U} \\ 0 & \text{if } A \notin \mathcal{U}. \end{cases}$$

Moreover, the set of purely finitely additive zero-one measures is precisely the set of free ultrafilters on $\mathbb{N}$.[13]

Ultrafilters can also be used to generalize the notion of convergence of real sequences. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded sequence $\{a_n\}$, we write $a = \lim_{\mathcal{U}} a_n$ if $\{j : \ |a_j - a| < \varepsilon\} \in \mathcal{U}$ for every positive ε. It can be shown that a unique such $a \in \mathbb{R}$ always exists and is an accumulation point of $\{a_n\}$. In this way, a free ultrafilter generalizes the standard limit operation.

[12]Some authors refer to a finitely additive measure as a charge (Bhaskara Rao and Bhaskara Rao, 1983).

[13]See Aliprantis and Border (1994) for more details.

For a set $A \in \mathbb{N}$, let $A(n)$ be the number of elements of A that are less than or equal to n. Formally, let $A(n) = |\, A \cap \{1, \ldots, n\}\, |$. We define the *asymptotic density* (or *natural density*) of A, denoted $d(A)$, by

$$d(A) = \lim_{n \to \infty} \frac{A(n)}{n},$$

if this limit exists.[14] For example, the asymptotic density of any finite set is 0 and the asymptotic density of the set of even numbers is 1/2. Asymptotic density provides us with the "intuitively correct" size of some infinite sets. For example, for any $m \in \mathbb{N}$, define $F_{i,m}$ to be the following arithmetic progression

$$F_{i,m} = \{i + mj : j \in \mathbb{N}_0\}$$

for $i = 1, \ldots, m$. Clearly, $d(F_{i,m}) = 1/m$, as one would expect. In addition, we define the *lower asymptotic density* of A by

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{A(n)}{n}$$

and the *upper asymptotic density* of A by

$$\bar{d}(A) = \limsup_{n \to \infty} \frac{A(n)}{n}.$$

Obviously, $d(A)$ exists if and only if $\underline{d}(A) = d(A) = \bar{d}(A)$.

Unfortunately, asymptotic density does not exist for all subsets of $\mathbb{N}$.[15] In fact, if A and B are non-disjoint subsets of $\mathbb{N}$ and $d(A)$ and $d(B)$ exist, then $A \cup B$ may not have asymptotic density.[16] Let $\mathcal{D} \subset P(\mathbb{N})$ be the collection of sets that that have asymptotic density. That is, $A \in \mathcal{D}$ if and only if $d(A)$ exists. For later use, we list two properties of asymptotic density (Niven

[14] Campbell and Kelly (1995) use asymptotic density to study social choice trade-offs with an infinite set of alternatives.

[15] Therefore asymptotic density is not a measure on $\mathbb{N}$.

[16] Therefore $\mathcal{D}$ is not even an Boolean algebra.

et al., 1991):

1. if $d(A)$ exists, then $d(A^c)$ exists and is equal to $1 - d(A)$, and

2. if A and B are disjoint subsets of $\mathbb{N}$ and $d(A)$ and $d(B)$ exist, then $A \cup B$ has asymptotic density $d(A) + d(B)$.

Because not all sets have density, it is natural to consider extensions of asymptotic density to all subsets of $\mathbb{N}$. We first establish some definitions. A *density measure* is a finitely additive measure on $\mathbb{N}$ that extends density. That is, μ is a density measure if it is a finitely additive measure on $\mathbb{N}$ such that $\mu(A) = d(A)$ whenever $d(A)$ exists. It is well known (Maharam, 1976; van Douwen, 1992) that a density measure on $\mathbb{N}$ exists. Indeed, there are many such measures, as the extension is not unique. One method of extension is to use the generalized limit operator $\lim_{\mathcal{U}}$, for some free ultrafilter $\mathcal{U}$, and define $\mu(A) = \lim_{\mathcal{U}} A(n)/n$. As discussed by Lauwers (1998a), all possible extensions are given by

$$\mu(A) = \int_{\Omega} \left[\lim_{\mathcal{U}} \frac{A(n)}{n} \right] d\varphi(\mathcal{U}), \tag{1}$$

where φ is a finite Borel measure on Ω, the space of all free ultrafilters on $\mathbb{N}$. From this it follows that for any density measure μ, $\underline{d}(A) \leq \mu(A) \leq \bar{d}(A)$.

4 Results

In this section, we consider the three versions of anonymity defined in section 2. We give results that illustrate the undesirability of strong and finite anonymity, and show that bounded anonymity is the most satisfactory condition of the three.

4.1 Strong Anonymity

We begin with the strong anonymity condition, which requires an aggregation rule to be invariant under any permutation of $\mathbb{N}$. A class of rules that satisfies this condition is the following:

Definition 6 *For $n \in \mathbb{N}_0 \cup \{\infty\}$, an aggregation rule f is a **n-rule** if*

$$f(V) = \begin{cases} 0 & \text{if } |V| < n \\ 1/2 & \text{if } |V| \geq n \text{ and } |V^c| \geq n \\ 1 & \text{if } |V^c| < n \end{cases}$$

for all $V \in P(\mathbb{N})$.

In other words, an n-rule chooses 0 or 1 if fewer than n voters disagree with the choice, and otherwise chooses 1/2. Thus, any preference profile in which both alternatives receive an infinite number of votes results in a tied outcome. We include in this definition the possibility of a 0-rule, which assigns value 1/2 to every subset of $\mathbb{N}$, and a ∞-rule, which assigns value 0 to every finite set, 1 to every cofinite set, and 1/2 to all other sets.

Our first theorem is that the class of n-rules are the only rules that satisfy neutrality, monotonicity, and strong anonymity.

Theorem 1 *An aggregation rule f satisfies neutrality, monotonicity, and strong anonymity if and only if f is a n-rule, for some $n \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof: The "if" statement is clearly true. To prove the converse, suppose that f satisfies neutrality, monotonicity, and strong anonymity. Begin by considering a set $V \in \mathcal{I}$. Obviously, $V^c \in \mathcal{I}$ and thus there exists a permutation π_{VV^c}. By construction, $\pi_{VV^c}V = V^c$. So by strong anonymity, $f(V^c) = f(V)$. But neutrality requires that $f(V^c) = 1 - f(V)$, so it follows that $f(V) = 1/2$.

It remains to deal with the finite and cofinite subsets of $\mathbb{N}$. As each finite set is the complement of a cofinite set, and vice versa, by neutrality it is sufficient to consider only the collection of finite subsets of $\mathbb{N}$. So let V be a finite set. It is easy to find an infinite superset of V with infinite complement. By monotonicity, then, $f(V) \leq 1/2$ for all finite sets.

If $f(V) = 1/2$ for all finite sets $V \in P(\mathbb{N})$, then f is a 0-rule. If $f(V) = 0$ for all finite sets $V \in P(\mathbb{N})$, then f is a ∞-rule. So assume that for some finite $W \in P(\mathbb{N})$, $f(W) = 1/2$ and for some finite $W' \in P(\mathbb{N})$, $f(W') = 0$. Let $n = \min\{|V| : f(V) = 1/2\}$. By strong anonymity, if $V \in P(\mathbb{N})$ is such that $|V| = n$, then $f(V) = 1/2$. If $Y \in P(\mathbb{N})$ is such that $|Y| > n$, then there exists a set $V \subset Y$ with $|V| = n$. By monotonicity, $f(Y) = 1/2$. By the definition of n, if $Z \in P(\mathbb{N})$ is such that $|Z| < n$, then $f(Z) = 0$. Thus f is a n-rule. ∎

It is immediate from the definition that every n-rule assigns 1/2 to every $V \in \mathcal{I}$. So by imposing strong anonymity, we lose any ability to differentiate between infinite sets of votes.[17] Every ballot in which both alternatives receive an infinite number of votes must be a tied outcome. Moreover, this is the typical case as almost all ballots in $P(\mathbb{N})$ are of this type. Thus, strong anonymity is too strong a condition to impose.

If we strengthen the monotonicity condition by requiring positive responsiveness, we have the following:

Corollary 1 *There is no aggregation rule that satisfies neutrality, positive responsiveness, and strong anonymity.*

This follows immediately from that fact that for two sets $A, B \in \mathcal{I}$, such that $A \subsetneq B$, the conditions of Theorem 1 require $f(A) = f(B) = 1/2$. This must also be true for any rule that satisfies the stronger condition of positive responsiveness. But this fact is obviously a violation of positive responsiveness.

[17]Lauwers (1998b) also makes this point about strong anonymity.

4.2 Finite Anonymity

In this part we consider a weakening of the anonymity axiom to permit permutations that operate on only a finite number of voters. While we do not provide a characterization, we give two examples that illustrate our claim that finite anonymity is too weak. First, we show that there exists a voting rule that satisfies finite anonymity (and monotonicity and neutrality) that chooses alternative 1 if the even voters support it, but does not choose it if the odd voters support it instead. Moreover, we show that there exists a voting rule for which an arbitrarily small "minority" holds power over an overwhelming majority.[18] From this we conclude that finite anonymity is not an appropriate property for majority rule in an infinite population.

We first show that a purely finitely additive zero-one measure satisfies finite anonymity, monotonicity, and neutrality.[19]

Theorem 2 *Suppose an aggregation rule f is a purely finitely additive zero-one measure. Then f satisfies neutrality, monotonicity, and finite anonymity.*

Proof: As f is a purely finitely additive zero-one measure, there exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that

$$f(V) = \begin{cases} 1 & \text{if } V \in \mathcal{U} \\ 0 & \text{if } V \notin \mathcal{U}. \end{cases}$$

Referring back to Definition 5, f satisfies neutrality by part 4 and f satisfies monotonicity by part 3. To see that f satisfies finite anonymity, take $A \in P(\mathbb{N})$ and a finite permutation π, and let $B = \pi A$. As π is finite, the sets A and B differ by only a finite number of elements.

Suppose that $f(A) = 1$, so $A \in \mathcal{U}$. Define $C = A \cup B$ and $D = C \setminus B$. As $A \subseteq C$, $C \in \mathcal{U}$, but $D \notin \mathcal{U}$ because D is finite. By construction,

[18] This is similar to the work of Fishburn (1970) and Kirman and Sondermann (1972) on Arrow's Theorem.

[19] Lauwers and Van Liedekerke (1995) and Mihara (1999) show that an aggregation rule based on a free ultrafilter satisfies finite anonymity.

$D \cup B = C \in \mathcal{U}$, so by part 1 of Lemma 1, $B \in \mathcal{U}$ and thus $f(\pi A) = 1$. A symmetric argument applies to the case $f(A) = 0$. ∎

Fix an arbitrary purely finitely additive zero-one measure μ. Let $E = \{2, 4, 6, \dots\}$ denote the set of even-numbered voters and let $O = \{1, 3, 5, \dots\}$ denote the set of odd-numbered voters. As $E \cup O = \mathbb{N}$, without loss of generality we can suppose that $\mu(E) = 1$. Setting $f = \mu$ and appealing to the above theorem yields the following corollary:

Corollary 2 *There exists an aggregation rule f that satisfies neutrality, monotonicity, and finite anonymity such that $f(E) = 1$ and $f(O) = 0$.*

Intuitively, the sets E and O each contain "half" of the voters in the society and thus majority rule should assign a tie to the case in which these two groups vote against each other. Thus, this corollary illustrates the first problem with finite anonymity, namely that it admits rules that assign a winner to situations that ought to be tied.

The next corollary shows that the dubiousness of finite anonymity goes further than this first problem.

Corollary 3 *For any $\varepsilon > 0$, there exists an aggregation rule f that satisfies neutrality, monotonicity, and finite anonymity such that $f(L) = 1$ for a set $L \in P(\mathbb{N})$ with $d(L) < \varepsilon$.*

Proof: Let $\mathcal{U}$ be an arbitrary free ultrafilter on $\mathbb{N}$ and pick $m \in \mathbb{N}$ such that $1/m < \varepsilon$. Recall that $F_{i,m} = \{i + mj : j \in \mathbb{N}_0\}$ for $i = 1, \dots, m$ and $d(F_{i,m}) = 1/m$ for each i. Since $\cup_{i=1}^{m} F_{i,m} = \mathbb{N}$, by repeated application of part 1 of Lemma 1, we see that $F_{i,m} \in \mathcal{U}$ for some i. Setting $L = F_{i,m}$ and letting f be the purely finitely additive zero-one measure determined by $\mathcal{U}$, the result is established. ∎

The importance of this corollary is that we can make the set L "arbitrarily small" (though still infinite) in the sense of asymptotic density and yet the votes of the members of L still determine the outcome. In other words, finite anonymity permits rules in which an arbitrarily small minority is decisive against the much larger majority. Obviously, this is incompatible with our intuitive sense of preference aggregation by majority rule. Significantly, this corollary bears a strong resemblance to previous results on "invisible dictators" in the context of Arrow's Theorem with an infinite population (Fishburn, 1970; Kirman and Sondermann, 1972; Mihara, 1999). This connection stems from the use of free ultrafilter aggregation rules in both results.

4.3 Bounded Anonymity

Thus far, we have argued that strong anonymity is too strong because it cannot discriminate between infinite sets and, conversely, finite anonymity is too weak because it permits arbitrarily small minorities to overrule large majorities. We next consider bounded anonymity, which is an intermediate form of the axiom that requires invariance to similarly sized sets, as measured by asymptotic density.

Recall that in a finite population, neutrality, monotonicity, and anonymity are satisfied only by quota (or supermajority) rules. Often, quota rules are referred to as q-rules. As discussed in section 3, asymptotic density yields the intuitively correct sizes of many sets in $P(\mathbb{N})$ and so we can use this concept to define the family of *density q-rules* in an analogous fashion to q-rules in a finite population. However, asymptotic density does not exist for all subsets of $P(\mathbb{N})$ and so a density q-rule can only assign an outcome to sets in $\mathcal{D}$, the collection of sets with density. One way of dealing with this problem is to extend asymptotic density to measure all of the sets in $P(\mathbb{N})$, as described in section 3. Such an extension is called a density measure. By using a given density measure to judge the size of sets, we can define a family of *density measure q-rules*. Unlike density q-rules, a density measure q-rule assigns an outcome to every subset of $\mathbb{N}$. We formalize these notions in the following definition. It is complicated by the need to specify the outcome assigned to a set with measure equal to q.

Definition 7 *An aggregation rule f is*

1. *an open density measure q-rule (resp. open density q-rule) if there is a density measure μ (resp. $\mu = d$) and a value $q \in [1/2, 1]$ such that, for all $V \in P(\mathbb{N})$ (resp. $V \in \mathcal{D}$),*

$$f(V) = \begin{cases} 0 & \text{if } \mu(V) < 1-q \\ 1/2 & \text{if } \mu(V) \in [1-q, q] \\ 1 & \text{if } \mu(V) > q, \end{cases}$$

2. *a closed density measure q-rule (resp. closed density q-rule) if there is a density measure μ (resp. $\mu = d$) and a value $q \in (1/2, 1]$ such that, for all $V \in P(\mathbb{N})$ (resp. $V \in \mathcal{D}$),*

$$f(V) = \begin{cases} 0 & \text{if } \mu(V) \leq 1-q \\ 1/2 & \text{if } \mu(V) \in (1-q, q) \\ 1 & \text{if } \mu(V) \geq q, \end{cases}$$

3. *a density measure q-rule if it is either an open density measure q-rule or a closed density measure q-rule.*

4. *a density q-rule if it is either an open density q-rule or a closed density q-rule.*

It follows from Theorem 1 that if f is a n-rule, for some $n \in \mathbb{N}_0 \cup \{\infty\}$, then f satisfies neutrality, monotonicity, and bounded anonymity. Our interest in density measure q-rules is justified by the following result.

Theorem 3 *An aggregation rule f satisfies neutrality, monotonicity, and bounded anonymity if f is a n-rule, for some $n \in \mathbb{N}_0 \cup \{\infty\}$, or f is a density measure q-rule.*

Proof: Clearly, a n-rule satisfies the three axioms. A density measure q-rule obviously satisfies neutrality and monotonicity. To show that it also satisfies bounded anonymity it suffices to show that if μ is a density measure and $\pi \in \mathcal{G}$, then $\mu(\pi V) = \mu(V)$ for all $V \in P(\mathbb{N})$. From Blümlinger (1996, Lemma 2), $\pi \in \mathcal{G}$ implies that $\lim_{n\to\infty}(V(n) - \pi V(n))/n = 0$, for all $V \in P(\mathbb{N})$. Thus, for any free ultrafilter $\mathcal{U}$, $\lim_{\mathcal{U}}(V(n) - \pi V(n))/n = 0$. From equation (1), it follows that $\mu(\pi V) = \mu(V)$ for all $V \in P(\mathbb{N})$. ∎

Before we address the converse question, we present the following two lemmas. The first is that density is full valued (Maharam, 1976) and that any infinite set has a infinite subset with zero asymptotic density.

Lemma 2 *For each $B \in \mathcal{I}$, there exists a set $A \in \mathcal{I}$ such that $A \subseteq B$ and $d(A) = 0$. In addition, for each $B \in \mathcal{I}$ such that $d(B)$ exists, if $\alpha \in [0, d(B)]$, then there exists a set $A \in \mathcal{I}$ such that $A \subseteq B$ and $d(A) = \alpha$.*

Proof: The result is trivial for $\alpha = d(B)$. It is clear that if $d(A) = \alpha \in (0, 1)$, then $A \in \mathcal{I}$. Combining this with Obata (1988, Proposition 1.3) yields the desired result for $\alpha \in (0, d(B))$. If $\alpha = 0$, the result follows by the same argument as the proof of Lemma 4.11 of van Douwen (1992). ∎

Our second lemma states that, under bounded anonymity, sets with equal density must be assigned the same outcome.

Lemma 3 *Let $A, B \in \mathcal{I} \cap \mathcal{D}$ with $d(A) = d(B)$. Then, if f satisfies bounded anonymity, $f(A) = f(B)$.*

Proof: Suppose $A, B \in \mathcal{I} \cap \mathcal{D}$ with $d(A) = d(B)$. Then, as shown by Obata (1988, Lemma 2.4), there exists a permutation $\pi \in \mathcal{G}$ with $B = \pi A$. Bounded anonymity thus implies that $f(A) = f(B)$. ∎

We say that an aggregation rule f *agrees with* a density q-rule if it assigns the same outcome as some density q-rule to all sets $V \in \mathcal{D}$. Note that, by definition, all density measures assign the measure $d(A)$ to sets $A \in \mathcal{D}$ and thus density measure q-rules necessarily agree with a density q-rule. We next show that the converse of Theorem 3 holds in the sense that a rule satisfying the given conditions must be a n-rule or agree with a density q-rule. In either case, this result establishes the converse of Theorem 3 for sets with density.

Theorem 4 *An aggregation rule f satisfies neutrality, monotonicity, and bounded anonymity only if f is a n-rule, for some $n \in \mathbb{N}_0 \cup \{\infty\}$, or f agrees with a density q-rule.*

Proof: First suppose that $f(V) = 1/2$ for all $V \in \mathcal{I}$. The remaining sets are either finite or cofinite. If $A \in P(\mathbb{N})$ is finite, then $f(A) \neq 1$. To see this, suppose not. A^c is cofinite and $f(A^c) = 0$ by neutrality, but it is easy to find a cofinite set C that is image of A^c under a bounded permutation and that contains A as a subset. By anonymity, $f(C) = 0$, but by monotonicity, $f(C) = 1$. This contradiction establishes that $f(A) \neq 1$. Now suppose that $A, B \in P(\mathbb{N})$ are both finite. (By neutrality, this argument also applies to cofinite sets.) Then it is clear that there is a permutation $\pi \in \mathcal{G}$ with $A = \pi B$ if and only if $|A| = |B|$. Then by the same arguments as in the proof of Theorem 1, f must be a n-rule.

The other possibility is that $f(V) = 1/2$ does not hold for all $V \in \mathcal{I}$. We first suppose that $f(V) \neq 1/2$ for some $V \in \mathcal{I} \cap \mathcal{D}$. Let $O = \{1, 3, 5, \dots\}$ and $E = \{2, 4, 6, \dots\}$. Then, as $d(O) = d(E) = 1/2$, lemma 3 implies that $f(E) = f(O)$, while neutrality requires that $f(E) = 1 - f(O)$. Thus $f(E) = f(O) = 1/2$. Appealing to lemma 3 again, this implies that for any set C with asymptotic density equal to 1/2, $f(C) = 1/2$. So, the set V, for which $f(V) \neq 1/2$, must not have density equal to 1/2. Without loss of generality, suppose $f(V) = 0$. (If this does not hold for V, it will hold for V^c by neutrality.) By lemma 2, there exists $Z \subseteq V$ with $d(Z) = 0$ and $Z \in \mathcal{I}$. By monotonicity, $f(Z) = 0$. Therefore, lemma 3 implies that for any $Z' \in \mathcal{I}$ with $d(Z') = 0$, $f(Z') = 0$. Moreover, for any finite set F, it is possible to construct a set $Z' \in \mathcal{I}$ with $F \subset Z'$ and $d(Z') = 0$. Thus monotonicity implies that if F is finite, $f(F) = 0$. By neutrality, if C is cofinite, $f(C) = 1$.

We next establish that for all $A, B \in \mathcal{I} \cap \mathcal{D}$, $d(A) \geq d(B)$ implies $f(A) \geq f(B)$. Suppose not. That is, suppose there exists $A, B \in \mathcal{I} \cap \mathcal{D}$, such that $d(A) \geq d(B)$ and $f(A) < f(B)$. It cannot be the case that $d(A) = d(B)$, by lemma 3. So $d(A) > d(B)$. Then, by lemma 2, there exists $C \in \mathcal{I}$ such that $C \subseteq A$ and $d(C) = d(B)$. But then, again by lemma 3, $f(C) = f(B)$ and monotonicity requires that $f(A) \geq f(C) = f(B)$, a contradiction.

Now let

$$\gamma = \sup_{D \in \mathcal{D} \cap \mathcal{I}} \{d(D) : f(D) = 0\}.$$

This is well-defined as $f(Z) = 0$ for any Z with $d(Z) = 0$. Also, $\gamma \leq 1/2$ by lemma 2 and monotonicity. First, suppose that there exists $D \in \mathcal{D} \cap \mathcal{I}$ with $f(D) = 0$ and $d(D) = \gamma$. Then $\gamma < 1/2$. Now let $L \in \mathcal{D} \cap \mathcal{I}$ with $d(L) \leq \gamma$. By the result in the previous paragraph, $f(L) = 0$. By neutrality, $d(L) \geq 1 - \gamma$ implies $f(L) = 1$. By neutrality and the definition of γ, $\gamma < d(L) < 1 - \gamma$ implies $f(L) = 1/2$. That is, f agrees with a closed density q-rule with $q = 1 - \gamma$. On the other hand, if the supremum in the definition of γ is not obtained, then $\gamma > 0$ and a slight modification of this argument shows that f agrees with an open density q-rule with $q = 1 - \gamma$.

The final case to consider is that $f(V) = 1/2$ does not hold for all $V \in \mathcal{I}$, but it does hold for all $V \in \mathcal{I} \cap \mathcal{D}$. By the same argument as above, f must assign 0 to all finite sets and 1 to all cofinite sets. This means that f agrees with a open density 1-rule. ∎

It should be noted that this characterization specifies an outcome only for sets in $\mathcal{D}$. It does not require any particular outcome on sets without density. This immediately raises two questions to consider. First, how complete is this characterization? Specifically, how many sets does $\mathcal{D}$ contain, relative to the number of sets in $P(\mathbb{N})$? Second, can we strengthen this result to characterize the outcomes outside of $\mathcal{D}$?

The first question is easy to answer. The strong law of large numbers (Billingsley, 1986) implies that almost all subsets of $\mathbb{N}$ have asymptotic density equal to 1/2. Thus, Theorem 4 identifies an outcome for almost all sets V. Moreover, as every n-rule and density q-rule assigns a tied outcome to any set with density equal to 1/2, the following corollary is immediate:

Corollary 4 *Suppose an aggregation rule f satisfies neutrality, monotonicity, and bounded anonymity. Then $f(V) = 1/2$ for almost all V.*

In other words, a randomly chosen sequence of votes will, with probability one, be an element of $\mathcal{D}$ and be assigned a tie.

The second question posed above is what outcome should be assigned to sets without density. Given our work so far, it is natural to consider extensions of asymptotic density to all subsets of $\mathbb{N}$, that is, density measures, as a possible answer. Perhaps an aggregation rule satisfying the conditions in Theorem 4 must be describable as a density measure q-rule for all sets in $P(\mathbb{N})$. We now describe an example which shows that this is not true. In particular, we construct an aggregation rule that satisfies neutrality, monotonicity, and bounded anonymity but that is not a density measure q-rule for any q.

We begin the construction by assigning outcomes to sets in $\mathcal{D}$ according to an open density 2/3-rule. This fixes the value of f for all sets in $\mathcal{D}$. Now we turn to those sets not in $\mathcal{D}$. We say a set $A \in P(\mathbb{N})$ with $A \notin \mathcal{D}$ is *grounded by* a set $D \in \mathcal{D}$ if $D \subseteq A$ and $f(D) = 1$. With this definition, we partition the sets not in $\mathcal{D}$ into one of two collections, $\mathcal{M}$ and $\mathcal{F}$, according to the following rule: if either A or A^c is grounded by some set $D \in \mathcal{D}$, then both A and A^c are assigned to $\mathcal{M}$, and if neither are grounded, they are both assigned to $\mathcal{F}$. Next, we assign outcomes to sets in $\mathcal{M}$ according to the following simple rule: if A is grounded by some set $D \in \mathcal{D}$, then $f(A) = 1$; if not, then A^c must be grounded and so we assign $f(A) = 0$. Finally, we define $f(A) = 1/2$ for every $A \in \mathcal{F}$.

It is easy to show that the collection $\mathcal{M}$ is nonempty. That $\mathcal{F}$ is also nonempty follows from a result of Grekos (1978) that for any real α and β, $0 \leq \alpha < \beta \leq 1$, there exists a set $A \in P(\mathbb{N})$ with $\underline{d}(A) = \alpha$ and $\bar{d}(A) = \beta$ such that any $B \subset A$ with $B \in \mathcal{D}$ must have $d(B) = 0$.[20]

By construction, f satisfies neutrality. Once we note that, by construction, no set in $\mathcal{F}$ contains a subset in $\mathcal{D}$ or $\mathcal{M}$ assigned a outcome of 1, it is clear that f also satisfies monotonicity. All that remains is to show that f satisfies bounded anonymity. To see this, let $\pi \in \mathcal{G}$ and $A \in \mathcal{D}$. It follows from the arguments in the proof of Theorem 3 that $\pi A \in \mathcal{D}$ and $d(A) = d(\pi A)$. Thus, bounded anonymity does not link the outcomes of sets with and without density. The only other possible violation of anonymity is if there is a permutation $\pi \in \mathcal{G}$ and sets $A \in \mathcal{M}$ and $B \in \mathcal{F}$ such that $\pi A = B$. By definition, either A or A^c is grounded by a set $D \in \mathcal{D}$. Since $\pi A^c = B^c$ holds, we can suppose without loss of generality that A is grounded by a set $D \in \mathcal{D}$. Letting $E = \pi D$, $E \in \mathcal{D}$ and $d(D) = d(E)$ must hold, as just mentioned. Because $f(D) = 1$, $d(D) > 2/3$, so $f(E) = 1$ as well. But $\pi A = B$ and $D \subseteq A$ imply that $E \subseteq B$, which contradicts $B \in \mathcal{F}$. Thus f satisfies bounded anonymity.

To complete our argument, we show that f is not representable as a density measure q-rule. As all density measures agree on all sets in $\mathcal{D}$, we are restricted to density measure 2/3-rules. As mentioned above, we can find a set $A \in \mathcal{F}$ with $2/3 < \underline{d}(A) < \bar{d}(A)$. Because $\underline{d}(A) \leq \mu(A) \leq \bar{d}(A)$ for every density measure μ, no density measure 2/3-rule can assign $f(A) = 1/2$, as required. Therefore, f is not a density measure q-rule. This shows that the converse of Theorem 3 does not hold and so we are unable give a complete characterization of these rules.

We conclude this section by reconsidering May's positive responsiveness axiom. This axiom requires that a change in a single voter's preference must break a tie. This is clearly a strong requirement in finite populations and

[20] See also Grekos and Volkmann (1987).

it is made even stronger in an infinite population in which any single voter has zero weight. Because of this, we propose a variation of this axiom that requires a tie be broken only if a positive fraction of voters (as measured by asymptotic density) change their preference.

Definition 8 *An aggregation rule f satisfies* ***density positive responsiveness*** *if f satisfies monotonicity and, for all $A \in \mathcal{D}$ with $f(A) = 1/2$ and all $D \in \mathcal{D}$ with $A \cap D = \emptyset$ and $d(D) > 0$, $f(A \cup D) = 1$.*

In addition, we say f is a density majority rule if

$$f(V) = \begin{cases} 0 & \text{if } \ d(D) < 1/2 \\ 1/2 & \text{if } \ d(D) = 1/2 \\ 1 & \text{if } \ d(D) > 1/2, \end{cases}$$

for all $D \in \mathcal{D}$. That is, a density majority rule is an open density 1/2-rule. Moreover, any open density measure 1/2-rule agrees with density majority rule.

With these definitions, we can now prove an analogue of May's Theorem for infinite populations. It is clear that any open density measure 1/2-rule satisfies neutrality, density positive responsiveness, and bounded anonymity. In the following corollary, we show that the converse holds in the sense that any rule satisfying these three conditions must agree with density majority rule.

Corollary 5 *An aggregation rule f satisfies neutrality, density positive responsiveness, and bounded anonymity only if f agrees with density majority rule.*

Proof: As density positive responsiveness is a strengthening of monotonicity, we need only consider the rules identified by Theorem 4. Clearly, n-rules violate density positive responsiveness. If f agrees with a density q-rule with

$q > 1/2$, then take $E = \{2, 4, 6, \ldots\}$ and a set $L \in \mathcal{D}$ with $E \cap L = \emptyset$ and $d(L) < q - 1/2$. Then $E \cup L \in \mathcal{D}$ and $f(E \cup L) = 1/2$, which violates density positive responsiveness. The only possibility remaining is that f agrees with density majority rule. As this rule clearly satisfies density positive responsiveness, the result holds. ∎

Once again, we are unable to completely characterize our version of majority rule in infinite settings. While we can only establish the result for $\mathcal{D}$, almost all subsets of $\mathbb{N}$ belong to this collection of sets. Moreover, the example given above can be modified to show that this is the best possible result along these lines.

5 Conclusion

In this paper, we investigate majority rule with an infinite number of voters. This setting creates some interesting technical questions whose answers turn out to be quite subtle. We reject two forms of anonymity and settle on bounded anonymity as the appropriate axiom to use in this work. Under this condition, the notion of asymptotic density insures that sets with density get assigned the intuitively correct size. This allows us to give a characterization of q-rules for almost all sets.

Several avenues for further research remain open. First, although we only consider voters with strict preferences, we conjecture that similar results will hold if voter indifference is permitted. In this case, the concept of relative density is likely to play an important role. Formally, if $A \subseteq B \in P(\mathbb{N})$, then the relative density of A in B is defined as

$$d(A, B) = \lim_{n \to \infty} \frac{A(n)}{B(n)},$$

if this limit exists. If we let B be the set of voters that are not indifferent and let A be the set of voters who prefer alternative 1, then it seems likely that results similar to those given here would hold by replacing asymptotic density with relative density of A in B.[21]

Another natural generalization worth considering is sets of three or more alternatives and, specifically, the characterization of voting rules in a setting with an infinite number of voters. The same types of problems that we have discussed here would also occur with scoring rules such as Borda rule, approval voting, and negative voting.[22] It seems likely that the tools used in the present paper would aid in addressing these rules as well.

Finally, it may be productive to analyze other specifications of anonymity or monotonicity. For example, rather than focusing on groups of permutations, one could instead directly define equivalence classes on $P(\mathbb{N})$ and require an aggregation rule be invariant on all sets within each class. One example of this would be requiring invariance on sets for which the symmetric difference between them has density zero. As with Corollary 5, such changes would also give rise to new version of positive responsiveness. We leave this to further work.

[21] Alternatively, one could specify the unique order-preserving bijection from B onto $\mathbb{N}$ and use asymptotic density of the image of A under this mapping.

[22] But see Pazner and Wesley (1978) for a variant of plurality rule in a countably infinite society.

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