An Improved, Simple Construction of Many Halving Edges

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Abstract. We construct, for every even n, a set of n points in the plane that generates $\Omega\big(ne^{\sqrt{\ln 4}\cdot\sqrt{\ln n}}/\sqrt{\ln n}\big)$ halving edges. This improves Tóth's previous bound by a constant factor in the exponent. Our construction is significantly simpler than Tóth's.

1. Introduction

Let S be a set of n points in the plane in general position (no three collinear). A *k-edge* of S is a directed segment $\overrightarrow{xy}$, $x, y \in S$, such that exactly k points of S lie strictly on the left-hand side of the directed line determined by $\overrightarrow{xy}$. If n is even, then a *halving edge* of S is an undirected segment $\overline{xy}$, such that $\overrightarrow{xy}$ and $\overrightarrow{yx}$ are $\left(\frac{n-2}{2}\right)$-edges of S.

We can generalize these notions to higher dimensions. If S is a set of n points in general position in $\mathbb{R}^d$ (no $d+1$ points on a common hyperplane), then a *k-facet* of S is an oriented $(d-1)$-dimensional simplex with vertices $x_1, \ldots, x_d \in S$, such that exactly k points of S lie strictly on the positive side of the simplex. If $n-d$ is even, then a *halving facet* of S is an unoriented $\left(\frac{n-d}{2}\right)$-facet of S.

A related notion is that of a *k-set*. If S is a set of n points in $\mathbb{R}^d$, then a k-point subset $P \subseteq S$ is called a *k-set* of S if there exists an open half-space γ such that $P = S \cap \gamma$.

The notions of k-sets and k-facets are closely related. In the plane there is a one-to-one correspondence between the k-sets and the $(k-1)$-edges of S. In $\mathbb{R}^3$ the number of k-sets of S is given by

$$\big(g_{k-1}(S) + g_{k-2}(S)\big)/2 + 2,$$

where $g_j(S)$ denotes the number of j-facets of S [**3**]. There also exists a general relation in $\mathbb{R}^d$, though it is less direct (see Chapter 11 of [**9**]).

It is a long-standing open problem to determine the maximum number of k-sets of an n-point set in $\mathbb{R}^d$ (the dimension d is usually taken to be a constant, while n and k can be arbitrarily large). Even in dimension 2 there is a wide gap between

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the best known lower and upper bounds, and the gaps get larger as the dimension grows.

The current lower bound for the maximum number of halving edges in the plane is of the form

$$\Omega\big(ne^{c\sqrt{\ln n}}\big), \qquad c \text{ constant}, \tag{1.1}$$

and it is due to Tóth [**12**]. Tóth showed how to get $c = 0.282$, and claimed (without proof) that c can be improved to 0.744.

In this paper we improve the constant c in the bound (1.1). We show how to construct, for every even n, a planar n-point set that generates

$$f(n) = \Omega\big(ne^{\sqrt{\ln 4}\cdot\sqrt{\ln n}}/\sqrt{\ln n}\big) \tag{1.2}$$

halving edges. Note that $f(n) = \omega\big(ne^{c\sqrt{\ln n}}\big)$ for every constant $c < \sqrt{\ln 4} = 1.1774\ldots$. Although our construction follows the same general approach as Tóth's, it is significantly simpler.

The current upper bound for halving edges is $O(n^{4/3})$, due to Dey [**5**] (see also [**3**]). Erdős et al. [**8**] conjecture that the true bound for halving edges is $o(n^{1+\varepsilon})$ for every constant $\varepsilon > 0$.

Bounds in higher dimensions. There exists a general, non-trivial upper bound of the form $O(n^{d-c_d})$ for the number of halving facets of an n-point set $S \subset \mathbb{R}^d$ [**2, 4, 13**]. The proof uses fairly complex tools from combinatorics and algebraic topology. Unfortunately, the constants c_d tend very quickly to zero as d increases.

There are better upper bounds, with more elementary proofs, for the specific cases $d = 3$ and $d = 4$. They are $O(n^{2.5})$ [**11**] and $O(n^{4-2/45})$ [**10**], respectively.

The best lower bounds for halving facets in higher dimensions are derived from the planar lower bound. Starting from a construction for $\Omega(ng(n))$ halving edges in the plane, one obtains a construction for

$$\Omega(n^{d-1}g(n)) \tag{1.3}$$

halving facets in $\mathbb{R}^d$ [**6, 12**] (assuming $g(n) = \Theta(g(n/3))$, which is the case if $g(n)$ is "well-behaved", e.g., if $g(n)$ is as given in (1.1), (1.2)).

***k*-sensitive bounds.** So far we have discussed bounds for halving facets, which depend only on n. There are more general bounds for k-facets, which depend on both n and k.

An upper bound of $O(n^{d-c_d})$ for halving facets in $\mathbb{R}^d$ implies an upper bound of

$$O\big(n^{\lfloor d/2\rfloor}k^{\lceil d/2\rceil - c_d}\big)$$

for k-facets in $\mathbb{R}^d$, for all $1 \le k \le (n-d)/2$ [**1**]. This gives the asymptotically best k-sensitive upper bound known for every d.

Regarding lower bounds, it is not hard to construct an n-point set in $\mathbb{R}^d$ with

$$\Omega\big(n^{\lfloor d/2\rfloor}k^{\lceil d/2\rceil - 1}\big)$$

k-facets, $1 \le k \le (n-d)/2$ [**9**, p. 267]. Alternatively, the lower bound (1.3) implies a lower bound of

$$\Omega(nk^{d-2}g(k))$$

for k-facets in $\mathbb{R}^d$ (assuming $g(k) = \Theta(g(2k))$; see [**6, 12**]).

An excellent survey of the topic of k-sets is Chapter 11 of [**9**].

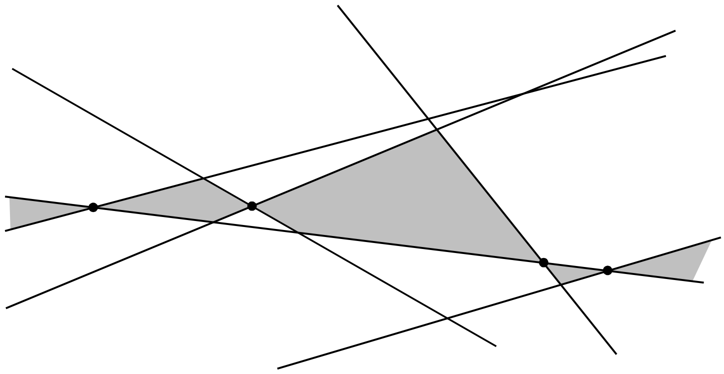

FIGURE 1. Middle-level cells and vertices of an arrangement of lines.

The dual setting. Following Matoušek [**9**], in this paper we work in the dual setting, since it is more convenient for our purposes.

Given a non-vertical line $l = \{\, (x, y) \in \mathbb{R}^2 \mid y = ax - b \,\}$, the *dual* of l is the point $l^* = (a, b) \in \mathbb{R}^2$. And given a point $p = (c, d) \in \mathbb{R}^2$, the *dual* of p is the (non-vertical) line $p^* = \{\, (x, y) \in \mathbb{R}^2 \mid y = cx - d \,\}$. A point p lies above (on, below) a non-vertical line l if and only if l^* lies above (on, below) p^*.

If L is a finite set of lines in the plane, then the *arrangement* $\mathcal{A}(L)$ is the decomposition of $\mathbb{R}^2$ induced by L into relatively open connected regions of dimensions 0, 1, and 2, called *vertices*, *edges*, and *cells* respectively. If no line of L is vertical, then the *level* of a point $p \in \mathbb{R}^2$ with respect to L equals the number of lines of L lying strictly below p. The level of an edge or cell of $\mathcal{A}(L)$ is the level of every point in that edge or cell.

Let $S = \{p_1, \ldots, p_n\}$ be a planar point set, and let $L = \{p_1^*, \ldots, p_n^*\}$ be the set of their dual lines. A k-edge of S corresponds uniquely to a vertex in $\mathcal{A}(L)$ at level k or level $n - k - 2$. A k-set $P \subseteq S$ corresponds uniquely to a cell in $\mathcal{A}(L)$ at level k or level $n - k$, except when P can be separated from the rest of S by a vertical line, in which case it corresponds to two unbounded cells in $\mathcal{A}(L)$, one at level k and one at level $n - k$.

Suppose n is even. Then a halving edge of S corresponds to a vertex of $\mathcal{A}(L)$ at the middle level, i.e. at level $(n - 2)/2$. Further, the middle-level cells of $\mathcal{A}(L)$ touch each other at their left and right endpoints, and these endpoints are the middle-level vertices of $\mathcal{A}(L)$ (Figure 1).

In Section 2 we construct, for every even n, a set L of n non-vertical lines in the plane (no three lines concurrent) whose arrangement $\mathcal{A}(L)$ contains $f(n)$ vertices at the middle level, with $f(n)$ as given in (1.2). This set L is the dual of a planar n-point set S that generates $f(n)$ halving edges.

2. The construction

We first build a sequence $L_0, L_1, L_2, \ldots$ of sets of non-vertical lines, such that the arrangement $\mathcal{A}(L_m)$ contains $f(|L_m|)$ middle-level vertices, with f as given in (1.2).

Each configuration L_m contains lines of two types: *plain* and *bold*. For each L_m we also construct a set V_m of middle-level vertices of $\mathcal{A}(L_m)$, such that each

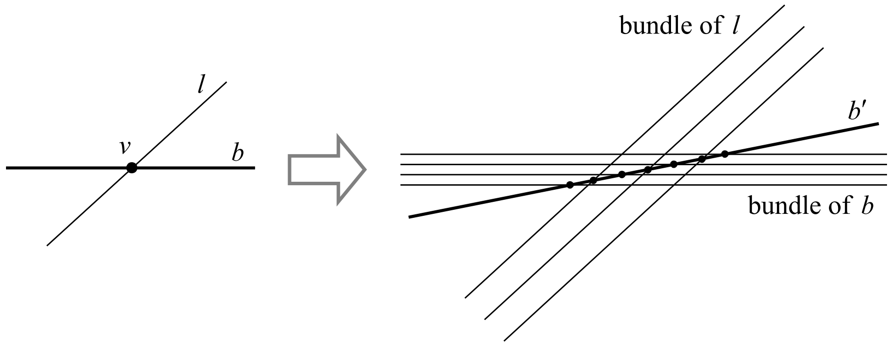

FIGURE 2. Constructing L_{m+1} from L_m.

$v \in V_m$ lies at the intersection of a plain line and a bold line. (V_m does not necessarily contain all the middle-level vertices of $\mathcal{A}(L_m)$.)

Our construction is inductive, and it depends on a sequence of free parameters $a_0, a_1, a_2, \ldots$, which we will set at the end.

The base case L_0 consists of two intersecting lines, one plain and one bold, and the corresponding vertex set V_0 contains the single intersection of these lines.

The inductive step. Given L_m and V_m, we construct L_{m+1} and V_{m+1} as follows. Each plain line $l \in L_m$ is replaced by a *bundle* of a_{m+1} parallel[1] *plain* lines separated by a sufficiently small distance ε_m. Each bold line $b \in L_m$ is replaced by a bundle of $a_{m+1}+1$ parallel *plain* lines with even smaller separation $\delta_m \ll \varepsilon_m$.

Thus, for each vertex $v \in V_m$, lying on plain line $l \in L_m$ and bold line $b \in L_m$, we obtain a grid $G = G_v$ of $(a_{m+1}+1) \times a_{m+1}$ plain-line intersections. If ε_m and δ_m are chosen small enough, then no two grids will overlap.

Through each such grid G we pass a new bold line b' with an intermediate slope between the slopes of the original lines b and l. The line b' crosses the grid diagonally through its center, alternately crossing the plain lines in the bundle of b and those in the bundle of l (Figure 2).

We choose δ_m sufficiently small compared to ε_m, so that b' is almost parallel to the bundle of b. More precisely, we make sure that for every other grid G' not lying on the bundle of b, line b' passes on the *same* side of G' as the bundle of b.

Each bold line $b' \in L_{m+1}$ generates $2a_{m+1}+1$ vertices within its corresponding grid G, as shown in Figure 2. We include all these vertices in V_{m+1}.

Correctness of the construction. To prove that our construction is correct, we need to show that all vertices in V_{m+1} are indeed middle-level vertices. We will in fact prove a stronger property about these vertices.

If v is a vertex of $\mathcal{A}(L_m)$, then we say that v is *strongly balanced* if the number of plain lines that pass above v equals the number of plain lines passing below v, and the number of bold lines passing above v equals the number of bold lines passing below v. (Note that v must lie at the middle level to be strongly balanced.)

LEMMA 2.1. *For all m, all vertices of V_m are strongly balanced.*

[1]It is sufficient for the construction that the lines be nearly parallel, so we can slightly perturb the lines and achieve general position.

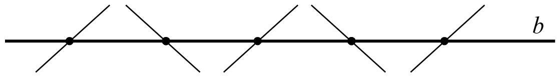

FIGURE 3. A bold line in L_m with its vertices from V_m. All these vertices lie within a single grid constructed in the inductive step from L_{m-1}.

PROOF. The claim is clearly true for V_0.

Suppose by induction that the claim is true for V_m. Note that each bold line $b \in L_m$ contains exactly $2a_m + 1$ vertices of V_m (letting $a_0 = 0$). Moreover, the plain lines of these vertices alternate between having a larger slope and a smaller slope than b (see Figure 3).

When building L_{m+1} from L_m, we replace each plain line in L_m by a_{m+1} parallel plain lines, and each bold line by $a_{m+1} + 1$ parallel plain lines plus $2a_m + 1$ bold lines almost parallel to them. We refer to this property as *uniform replacement*.

Now, consider a vertex $v \in V_m$, lying on plain line $l \in L_m$ and bold line $b \in L_m$. Let G be the grid in L_{m+1} corresponding to v, and let $b' \in L_{m+1}$ be the bold line that crosses through G. Let w be some vertex of V_{m+1} within G lying on b'.

Let $v_1, \dots, v_{2a_m}$ be the $2a_m$ vertices of V_m, besides v, that lie on b, and let $b'_1, \dots b'_{2a_m}$ be their corresponding bold lines in L_{m+1}.

Partition the lines of L_{m+1} into three sets. The first set S_1 includes all lines that originate from lines in L_m other than b and l. The original vertex $v \in V_m$ is strongly balanced by assumption, so by the uniform replacement property it follows that w is strongly balanced with respect to the lines in S_1.

The set S_2 includes all the lines that pass through the grid G, namely the plain lines in the bundles of b and l, and the bold line b'. It is clear that w is also strongly balanced with respect to the lines in S_2 (see Figure 2).

Finally, S_3 includes the bold lines $b'_1, \dots, b'_{2a_m}$. It can be verified that half of these lines pass above the grid G and the other half pass below G. Therefore, vertex w is also strongly balanced with respect to the lines in S_3.

Therefore, vertex w is strongly balanced, as claimed. □

Setting the free parameters. Let $n_m = |L_m|$ and $f_m = |V_m|$. We wish to choose appropriately the free parameters a_i, $i \geq 1$, and to express f_m, the number of middle-level vertices, as a function of n_m, the number of lines.

In our construction, each vertex in V_m is replaced by $2a_{m+1} + 1$ vertices in V_{m+1}. Therefore,

$$f_{m+1} = (2a_{m+1} + 1)f_m.$$

Now let us find n_{m+1}, the number of lines in L_{m+1}. First note that the number of bold lines in L_i equals the number of vertices in V_{i-1} for all $i \geq 1$. Therefore, L_{m+1} has

$$a_{m+1}(n_m - f_{m-1}) + (a_{m+1} + 1)f_{m-1} = a_{m+1}n_m + f_{m-1}$$

plain lines and f_m bold lines, so

$$n_{m+1} = a_{m+1}n_m + f_m + f_{m-1}.$$

We choose $a_i = 2^i$, for $i \geq 1$. (This is the optimal choice of the form $a_i = a^i$, $i \geq 1$; we omit the proof.) It follows that

$$f_m = f_0 \prod_{i=2}^{m+1} (2^i + 1) = \Theta\big(2^{(m^2+3m)/2}\big), \tag{2.1}$$

and

$$\begin{aligned} n_m &= 2^m n_{m-1} + \Theta(f_{m-1}) \\ &= 2^m n_{m-1} + \Theta\big(2^{(m^2+m)/2}\big). \end{aligned}$$

Let us analyze the sequence n'_m given by

$$n'_m = 2^m n'_{m-1} + k\, 2^{(m^2+m)/2} \qquad \text{for } m \geq 1,$$

where k is some constant. It follows by induction that

$$n'_m = 2^{(m^2+m)/2}(n'_0 + mk),$$

so

$$n_m = \Theta\big(m \cdot 2^{(m^2+m)/2}\big). \tag{2.2}$$

Dividing (2.1) by (2.2),

$$f_m/n_m = \Theta(2^m/m). \tag{2.3}$$

From (2.2) we get $m = \sqrt{2 \log_2 n_m} - \Theta(1)$. Finally, substituting into (2.3),

$$f_m = \Theta\big(n_m e^{\sqrt{\ln 4}\cdot\sqrt{\ln n_m}} / \sqrt{\ln n_m}\big). \tag{2.4}$$

Thus, we have achieved the bound (1.2) for n of the form $n = n_m$, $m \geq 1$. We can easily "fill in the gaps" and achieve (1.2) for all n, as in [**12**].

3. Discussion

Of course, the main challenge is to tighten the bounds on the maximum number of halving edges in the plane. Other open problems are:

(1) Derive more direct (and hopefully stronger) lower bounds for halving facets in $d \geq 3$ dimensions (rather than just "lifting" the 2-dimensional bound).
(2) Improve the lower bound for the maximum number of halving edges of a *dense* point set in the plane. A planar n-point set S is *dense* if the ratio between the largest distance and the smallest distance between any pair of points in S is $\Theta(\sqrt{n})$. The current lower bound for halving edges for a dense point set is $\Omega(n \log n)$. It is also known that an upper bound of $O(n^{1+c})$ for arbitrary sets implies an upper bound of $O(n^{1+c/2})$ for dense sets; see [**7**].

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