

# PROBABILITY CALCULATIONS UNDER THE IAC HYPOTHESIS

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ABSTRACT. We show how powerful algorithms recently developed for counting lattice points and computing volumes of convex polyhedra can be used to compute probabilities of a wide variety of events of interest in social choice theory. Several illustrative examples are given.

## 1. INTRODUCTION

Much research has been undertaken in recent decades with the aim of quantifying the probability of occurrence of certain types of election outcomes for a given voting rule under fixed assumptions on the distribution of voter preferences. Most prominent among these outcomes of interest are the so-called voting paradoxes, which have been shown to be unavoidable, hence the interest in how commonly they may occur. The survey [13] discusses these questions and gives a summary of results up to 2002.

In very many cases, particularly under the IAC hypothesis on voter preferences, the calculations involved amount simply to counting integer lattice points inside convex polytopes. In the social choice literature, two main methods have been used to carry out such computations. The first, dating back several decades, decomposes the polytope into smaller pieces each of which can be treated by elementary methods involving simplification of multiple sums. This method works fairly well for simple problems but requires considerable ingenuity and perseverance to carry out even for moderately complicated ones. More recently, more powerful methods have been introduced in [14, 12] but there are several recent instances where even these methods did not suffice to solve natural questions about 3-candidate elections.

The purpose of the present paper is to point out that there is an established mathematical theory of counting lattice points in convex polytopes (and the closely related issue of computing the volume of such a region), which has been partially rediscovered by workers in social choice theory. The area has recently been the subject of active research (see [4] for a good summary). Several more efficient new algorithms have been devised and implemented in publicly available software.

We aim to apply these new methods to answer questions in voting theory that have proven beyond the reach of previous authors. In addition we corroborate, correct, and unify the derivation of some previously published results by using this methodology. We believe that the solution of many hitherto difficult problems can now be relegated to a trivial computation. This should open the way for social choice theorists to tackle more difficult and realistic problems. We note that by Lepelley, Louichi and Smaoui [20] have recently, and independently from us, circulated a preprint with a similar goal, which covers very similar ground. The fact that two groups of researchers discovered this approach almost simultaneously shows that the time has indeed come for these methods to be assimilated by the social choice community.

The basic idea is that many sets of voting situations that are of interest can be characterized by linear equations and inequalities. The variables are usually the numbers of voters with each of the  $m!$  possible preference orders, where  $m$  is the number of alternatives. The set of such (in)equalities defines a convex polytope in  $\mathbb{R}^d$  for some  $d$ , given by  $Ax \leq b$  for some matrix  $A$  (here  $d \leq m!$  and the inequality may be strict, because we may first use equality relations to eliminate variables and reduce dimension). Each lattice point will correspond to a voting situation in the desired set. The probability that a randomly chosen situation has property under consideration is therefore a straightforward ratio

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of lattice point counts. Dividing through by  $n$ , the total number of voters, yields a convex polytope  $P$ , independent of  $n$ , in  $\mathbb{R}^d$ . For a given number  $n$  of voters, the dilation  $nP$  describes the set of lattice points that we wish to enumerate.

## 2. COUNTING LATTICE POINTS IN CONVEX POLYTOPES

We give only a brief description here. For more information we recommend [4].

The **Ehrhart series** of the rational polytope  $P$  is a rational generating function  $F(t) = P(t)/Q(t) = \sum_n a_n t^n$  whose  $n$ th Maclaurin coefficient  $a_n$  gives the number of lattice points inside the dilation  $nP$ . The function  $f : n \mapsto a_n$  is known to be a polynomial of degree  $d$  if all the vertices of  $P$  are integral; otherwise it is a quasipolynomial of some minimal period  $e$ . That is, the restriction of  $f$  to each fixed congruence class modulo  $e$  is a polynomial.

It is known that  $e$  is a divisor of  $m$ , where  $m$  is an integer such that all coordinates of vertices of  $mP$  are integers. The least such  $m$  is the least common multiple of the denominators of the coordinates of the vertices of  $P$  when each coordinate is written in reduced terms. However there are examples where  $e < m$  [22]. A method for determining  $e$  was presented in [14].

Many questions in voting theory are of most interest in the asymptotic case where  $n \rightarrow \infty$ . For small  $n$ , issues such as the method of tiebreaking used assume great importance, whereas in the limit such issues disappear (the situations in which ties occur correspond in the limit to the boundary of  $P$ ). We focus on limiting results in the present paper.

The leading coefficient of the quasipolynomial  $f$  is the same for all congruence classes: only the lower degree terms differ. It is well known that this leading coefficient is precisely the volume of  $P$ . For many purposes, knowledge of this coefficient is sufficient. The limiting probability under IAC as  $n \rightarrow \infty$  is simply the volume of  $P$  divided by the volume of  $X$  where  $X$  is the analogously defined polytope that describes all possible voting situations.

To compute the number of lattice points in  $nP$ , if that amount of detail is desired, we may use one of several algorithms. An attractive approach pioneered by Barvinok makes heavy use of rational generating functions; this is implemented in the software `LattE` [5, 15]. There are also several algorithms available for volume computation; see [1] for a survey of algorithms, a hybrid of which has been implemented in `vinci` [25] for floating point computation only. One of these algorithms has been used in the Maple package `Convex` [3], and uses exact rational arithmetic.

The software `LattE` gives the Ehrhart series as standard output. In order to extract the quasipolynomial formula for  $f(n)$  from the Ehrhart series, we may use interpolation. On each congruence class modulo  $e$ , we must evaluate  $f(n)$  at  $d+1$  distinct values of  $n$  in this class. Given the explicit expression  $F(t) = P(t)/Q(t)$  and a computer algebra system, such evaluations are trivially obtained (the  $a_n$  satisfy a linear recurrence relation with constant coefficients). The Lagrange interpolation formula then yields the desired formula for the particular polynomial that is applicable for the given congruence class.

Another (generally less efficient) method of extraction is to decompose  $F(t)$  into partial fractions. Note that  $F(0) = 1$  and we can arrange so that  $Q(t)$  factors as  $\prod_j (1 - \alpha_j t)$  for some complex numbers  $\alpha_j$ , possibly not distinct. We then have the partial fraction decomposition

$$F(t) = \sum_{\alpha} \sum_k c_{\alpha,k} (1 - \alpha t)^{-k}$$

where  $\alpha$  runs over the roots of  $Q$  and  $k$  runs from 1 to the multiplicity of  $\alpha$ .

This shows how the periodicity occurs: the factorization of  $Q(t)$  will introduce complex roots of unity and the terms corresponding to powers of these will simplify on each congruence class. In fact, on extracting the coefficient of  $t^n$  we obtain

$$[t^n]F(t) = \sum_{\alpha,k} \alpha^n c_{\alpha,k} \binom{n+k-1}{k-1}$$

and the terms  $\alpha^n$  simplify on each equivalence class modulo  $e$ . Note that  $e = 1$  (that is,  $f(n)$  is a single polynomial) if and only if  $Q$  factors completely over the rationals.

Note that since we know *a priori* that the coefficient of  $t^n$  is polynomially growing, all  $\alpha$  with  $|\alpha| \neq 1$  can be ignored, since their contribution must cancel (otherwise we would obtain terms exponentially growing or decreasing in  $n$ ). Unfortunately this observation does not help in the present case, because the Ehrhart series has a denominator of the form  $\prod_i (1 - t^{a_i})$ , so all the  $\alpha_j$  above are in fact roots of unity.

In summary, the Ehrhart series contains all information required to solve the problem of counting lattice points in polytopes parametrized by a single parameter  $n$ . The hardest step is usually determining the minimal period  $e$ .

### 3. EXAMPLES

In this section we compute, using the recipe above, a few probabilities under IAC that have been considered in the recent social choice literature. We emphasize problems where older methods have not yielded an answer, but also check results obtained by previous authors using older methods. Some of these earlier results appear to be incorrect. The use of a computer algebra system such as Maple [21] is essential for some of the more complicated examples.

**3.1. Manipulability.** We first consider the probability under IAC that a voting situation in a 3-candidate election is manipulable by some coalition. Counterthreats are not considered — we assume that some group of voters with incentive to manipulate will not be opposed by the other, naive, voters. See [24, 6] for more discussion of these (standard) assumptions.

For the classical rules plurality and antiplurality, the answer is known:  $7/24$  and  $14/27$  respectively [17, 18]. These results were derived by the earliest methods described above and required considerable hand computation. However, for the Borda rule, no such result has been derived even using more sophisticated methods. A good numerical approximation to the limit has been obtained. In [6] the authors used the method of [14] to obtain bounds on the solution but could not carry out the full computation. Using their method requires interpolation, hence computing the first  $6e$  coefficients of the Ehrhart series, where  $e$  is the minimal period of the quasipolynomial. They showed that  $e > 48$ , and since they computed these coefficients by exhaustive enumeration, it was not possible to carry out the computation to the end (the number of voting situations is of order  $n^5$ ). They estimated a value of 0.5025 for the limit.

However with more powerful tools the answers are easily obtained. We let  $n_1, \dots, n_6$  denote the number of voters with sincere preference order  $abc, acb, bac, bca, cab, cba$  respectively, and let  $x_i = n_i/n$ . Then  $\sum_i x_i = 1$  and  $x_i \geq 0$ . We use the linear systems derived for general positional rules in [24]. As shown in [24] we may assume without loss of generality that  $a$  wins,  $b$  is second, and  $c$  last in the election (this assumption will only affect lower order terms in our resulting quasipolynomial, and this is inevitable when different tie-breaking assumptions are made). Thus we must multiply our final answer by 6 since we are only considering one of the  $3!$  equally likely permutations of the candidates.

*Plurality.* We first consider the plurality rule. We define polytopes  $P_b, P_c, P_{bc}$  as follows. Consider the inequalities

$$(3.1) \quad 0 \leq x_1 + x_2 - x_3 - x_4 \quad (a \text{ beats } b \text{ (sincere)})$$

$$(3.2) \quad 0 \leq x_3 + x_4 - x_5 - x_6 \quad (b \text{ beats } c \text{ (sincere)})$$

$$(3.3) \quad 0 \leq -x_1 - x_2 + x_3 + x_4 + x_6 \quad (b \text{ beats } a \text{ (strategic)})$$

$$(3.4) \quad 0 \leq -x_1 - x_2 + 2x_3 + 2x_4 - x_5 + 2x_6 \quad (b \text{ beats } c \text{ (strategic)}).$$

The polytope  $P_b$  (the region where manipulation in favour of  $b$  is possible) is defined by the inequalities (3.1) – (3.4), the equality  $\sum_i x_i = 1$ , and the condition that all  $x_i$  are nonnegative. Polytope  $P_c$  is obtained by applying the permutation  $b \leftrightarrow c$ , which induces the permutation  $x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_5, x_4 \leftrightarrow x_6$ , and  $P_{bc} = P_b \cap P_c$  is just given by the union of the two sets of inequalities defining  $P_b$  and  $P_c$ .

The software `LattE` readily computes the Ehrhart series of each polytope. They are

$$H_b = \frac{12t^{12} + 24t^{11} + 44t^{10} + 56t^9 + 66t^8 + 64t^7 + 63t^6 + 44t^5 + 30t^4 + 14t^3 + 6t^2 + 2t + 1}{(1-t)^2(1-t^3)^4(1+t)^4(1+t^2)^3}$$

$$H_c = \frac{8t^{12} + 16t^{11} + 26t^{10} + 34t^9 + 38t^8 + 40t^7 + 41t^6 + 30t^5 + 20t^4 + 10t^3 + 4t^2 + 2t + 1}{(1-t^4)^3(1-t)^2(1-t^2)(1+t+t^2)^4}$$

$$H_{bc} = \frac{4t^8 + 5t^6 + 4t^5 + 4t^4 + 4t^3 + 2t^2 + 1}{(1-t)^4(1-t^4)^2(1+t+t^2)^4}$$

The series we require is therefore

$$H := H_b + H_c - H_{bc}$$

$$= \frac{16t^{12} + 32t^{11} + 57t^{10} + 68t^9 + 78t^8 + 74t^7 + 73t^6 + 50t^5 + 33t^4 + 14t^3 + 6t^2 + 2t + 1}{(1-t^4)^3(1-t)^2(1-t^2)(1+t+t^2)^4}.$$

Note that in order to factor the denominator of  $H$  completely we require both a cube root and fourth root of 1, hence a field extension of degree 12. Thus we expect the period of the quasipolynomial  $f(n) := [t^n]H(t)$  to be 12. We may determine the polynomial formula for  $f$  on each congruence class in more than one way, as described in section 2.

First, we try interpolation. Consider the polynomial expression valid for  $f(n)$  when  $n \equiv 0 \pmod{12}$ . This is a polynomial of degree 5 in  $n$ . We compute the values  $f(12j)$  for  $j = 0, \dots, 5$  and then determine the unique interpolating polynomial of degree 5 determined by these points, via, say, the Lagrange inversion formula. The built-in commands in Maple find this polynomial immediately: the answer is

$$f(n) = \frac{7}{17280}n^5 + \frac{1}{108}n^4 + \frac{3}{32}n^3 + \frac{15}{32}n^2 + \frac{137}{120}n + 1 \quad (n \equiv 0 \pmod{12}).$$

As a check, we substitute  $n = 96$  into this expression — the correct answer, namely  $[t^{96}]H(t) = 4176821$ , is obtained. Analogous formulae can be obtained in the same way for the other congruence classes modulo 12. For example, the result for  $n$  congruent to 6 modulo 12 is

$$f(n) = \frac{7}{17280}n^5 + \frac{1}{108}n^4 + \frac{3}{32}n^3 + \frac{15}{32}n^2 + \frac{61}{60}n + 5/8 \quad (n \equiv 6 \pmod{12}),$$

while that for  $n$  congruent to 1 is given by

$$f(n) = \frac{7}{17280}n^5 + \frac{1}{108}n^4 + \frac{341}{5184}n^3 + \frac{5}{36}n^2 - \frac{917}{17280}n - \frac{209}{1296} \quad (n \equiv 1 \pmod{12}).$$

Note that since the number of voting situations is given by  $\binom{n+5}{5} = (n+1)\cdots(n+5)/120$ , and we have only counted one-sixth of the manipulable situations, the limiting probability of manipulability is 720 times the leading coefficient of  $f$ , namely  $7/24$ . This agrees with the results obtained in [17] (note that the expressions for finite  $n$  do not agree, probably because of different tie-breaking assumptions yielding slightly different sets of manipulable voting situations).

As mentioned in section 2, another method would be to compute the full partial fraction decomposition of  $H$  over the extension field of  $\mathbb{Q}$  generated by a primitive 12th root of 1. This can be done easily by Maple. However the result is somewhat messy and the ensuing computation involving binomial coefficients is certainly no easier than using interpolation, so we omit it.

*Borda.* We now consider the Borda rule. We can attempt an analysis similar to the above (the polytopes are defined in a similar manner, and all coefficients lie in  $\{0, \pm 1, \pm 2, \pm 3\}$ ), but we run into serious complexity issues in this case.

The Ehrhart series  $F_b, F_c, F_{bc}$  given by `LattE` are such that when  $F := F_b + F_c - F_{bc}$  is simplified, its denominator is a product of cyclotomic polynomials (minimal polynomials for roots of unity). The corresponding roots of unity required are of orders whose least common multiple is 2520. So we are

still faced with the major task of computing  $e$ . It is still an open problem as to whether there exists an algorithm to determine  $e$  which runs in polynomial time (in the input size) when the dimension is fixed. A polynomial time algorithm to determine whether an integer  $p$  is equal to  $e$  was presented in [26], but has not been implemented in software as far as we are aware. Of course, we do not need to know the exact value of  $e$ , and we could assume it to be 2520. In order to determine exact formulae for  $f(n)$  in all cases by interpolation, we would require the first 15120 values of  $f(n)$ . Trying this in Maple we obtain an overflow error. However it would be possible in principle to compute these using the recurrence supplied by the rational form of  $F$ . We do not proceed further along these lines, but we indicate how the computation would go. Writing  $P(t) = \sum_k b_k t^k$ ,  $Q(t) = \sum_k c_k t^k$ ,  $F(t) = P(t)/Q(t) = \sum_n a_n t^n$  and comparing coefficients, we obtain  $b_n = \sum_{0 \leq k \leq n} c_k f(n-k)$ . This constant coefficient linear recurrence allows us to determine sequentially  $f(0), \dots, f(r)$  where  $r = \deg P$ , and for  $n > r$  we have the defining recurrence  $\sum_{0 \leq k \leq n} c_k f(n-k) = 0$ . In the present case  $\deg P = 75$  and  $\deg Q = 82$ , so the computation would be rather involved.

However, we can certainly determine the leading term of the quasipolynomial  $f$ , namely the volume of a certain region. It is convenient to eliminate  $x_6$  throughout, using the sole equality constraint  $\sum_i x_i = 1$ . In other words we look at the projection onto the subspace  $x_6 = 0$ . Since we are dividing by the volume of the projection of the simplex the exact scale factor is unimportant. This projection is defined by the conditions  $x_i \geq 0$  and  $\sum_{i=1}^5 x_i \leq 1$  — we call these the **standard inequalities**. The volume in  $\mathbb{R}^5$  of this simplex is easily computed to be  $1/5! = 1/120$ . Recalling the factor of 6 mentioned above, we shall therefore multiply the volume answer obtained below by 720 to compute the limiting probability.

The volume required is given by inclusion-exclusion as  $\text{vol}(R_b) + \text{vol}(R_c) - \text{vol}(R_b \cap R_c)$  where  $R_b, R_c$  respectively denote the region for which manipulation in favour of  $b$  or  $c$  is possible.

The conditions describing the sincere outcome reduce, after elimination of  $x_6$ , to

$$(3.5) \quad 2x_1 + 3x_3 + 2x_4 - x_5 \geq 1; \quad (a \text{ beats } b \text{ (sincere)})$$

$$(3.6) \quad 2x_1 + 3x_2 - x_4 + 2x_5 \geq 1; \quad (b \text{ beats } c \text{ (sincere)})$$

while the conditions describing the outcome after manipulation amount to

$$(3.7) \quad 3x_1 + 4x_2 + 3x_5 \leq 2 \quad (b \text{ beats } a \text{ (strategic)})$$

$$(3.8) \quad x_1 + 2x_2 + 2x_5 \leq 1 \quad (b \text{ beats } c \text{ (strategic)}).$$

Now  $R_b$  is defined by the standard inequalities and those in (3.5) – (3.8). Also  $R_c$  is obtained by applying the permutation  $b \leftrightarrow c$ , which induces the permutation  $x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_5, x_4 \leftrightarrow x_6$ , and  $R_{bc}$  is given by the union of the two sets of inequalities defining  $R_b$  and  $R_c$ .

The package **Convex** [3] immediately yields the answer when given this input. The respective volumes of  $R_b, R_c, R_{bc}$  are  $371/559872, 881/6531840, 170873/1714608000$  and the required limit is precisely  $132953/264600 \approx 0.5024678760$ .

The large denominators in the fractions above give a clue to the difficulty of this problem. **Convex** also computes the vertices of the polytope. The least common multiple of the denominators of the coordinates of the vertices is 72 for  $R_b$ , 504 for  $R_c$ , 1260 for  $R_{bc}$ . Thus the minimum period  $e$  is a divisor of  $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$ , as we already knew from above.

**3.2. Condorcet phenomena.** See the two surveys by Gehrlein [10, 11] for more information about previous work on this topic.

In [13] Gehrlein and Lepelley state “A very large number of studies (probably more than 50% of the studies that have been devoted to probability calculations in social choice theory) have been conducted to develop representations for the probability that Condorcet’s Paradox will occur, and for the Condorcet efficiency of various rules, with the assumptions of IC and IAC.”

*Condorcet’s paradox.* **Condorcet’s Paradox** occurs in a voting situation when there is no Condorcet winner — that is, no one candidate beats all others when only pairwise comparisons are considered.

$P C$	$A C$	$B C$	$(A \cap B) C$	$(A \cap P) C$	$(B \cap P) C$	$B (P \cap C)$	$B (A \cap C)$
0.88148 [9]	0.62963 [9]	0.91111[8]	0.61775	0.53040	0.81821	0.92282	0.98113

TABLE 1. (Joint) limiting Condorcet efficiencies of the standard positional rules under IAC, 3 candidates

This occurrence is independent of the voting rule being used. To compute its likelihood, we compute the complementary event.

Suppose that we have 3 alternatives  $a, b, c$ . Let  $C$  be the event that  $a$  is the Condorcet winner. This yields inequalities that boil down to

$$(3.9) \quad 2x_1 + 2x_2 + 2x_3 \geq 1 \quad (a \text{ beats } b \text{ pairwise});$$

$$(3.10) \quad 2x_1 + 2x_2 + 2x_5 \geq 1 \quad (a \text{ beats } c \text{ pairwise}).$$

Let  $P_C$  be the polytope defined by these and the standard inequalities. Then **Convex** yields  $\text{vol}(P_C) = 1/384$ , so that Condorcet's Paradox occurs with asymptotic probability  $1 - 3 \cdot 5!/384 = 1/16$  for IAC with 3 alternatives. This is of course a known result dating back several decades.

*Condorcet efficiency.* Similarly we may compute the **Condorcet efficiency** of a given rule, namely the conditional probability that it elects the Condorcet winner given that this winner exists. For a given scoring rule defined by weights  $(1, \lambda, 0)$ , let  $X_\lambda$  be the event that  $a$  is the winner when this rule is used. Clearly  $\Pr(X_\lambda) = 1/3$ .

These conditions describing  $X_\lambda$  amount to

$$(3.11) \quad x_1 + (1 + \lambda)x_2 + (2\lambda - 1)x_3 + (\lambda - 1)x_4 + 2\lambda x_5 \geq \lambda \quad (a \text{ beats } b \text{ with rule } \lambda)$$

$$(3.12) \quad 2x_1 + (2 - \lambda)x_2 + (1 + \lambda)x_3 + (1 - \lambda)x_4 + \lambda x_5 \geq 1 \quad (a \text{ beats } c \text{ with rule } \lambda).$$

The Condorcet efficiency of rule  $\lambda$  is  $\Pr(X_\lambda \cap C)/\Pr(C)$  which equals  $3 \cdot 5!(16/15) \text{vol}(P_\lambda \cap P_C)$ . In the special cases  $\lambda = 0, 1/2, 1$  of plurality, Borda, antiplurality, respectively, we obtain  $119/135, 41/45, 17/27$ . These last three results were obtained long ago by Gehrlein.

We can consider further intersections of such events. For example, Gehrlein has computed limiting results under IC for the conditional probability that rule  $\lambda$  chooses the Condorcet winner given that Borda does, that Borda does given that rule  $\lambda$  does, and that both rules choose the Condorcet winner given that it exists. The answers to these questions are easily found for IAC using the above methods and are listed in Table 1. These have not previously been published as far as we are aware. In Table 1 we let  $A|C$  denote the event that antiplurality chooses the Condorcet winner given that it exists,  $B|(P \cap C)$  the probability that Borda chooses the Condorcet winner given that plurality does, etc. These can be computed easily using the events  $C$  and  $X_\lambda$  above. For example, the entry  $B|(P \cap C)$  corresponds to the probability of the event that Borda and Condorcet agree given that plurality and Condorcet agree. This is simply the volume of the polytope  $P_{1/2} \cap P_C \cap P_0$  divided by the volume of  $P_0 \cap P_C$  (the factor of 3 cancels out because we are computing conditional probabilities via  $P(E_1|E_2) = P(E_1 \cap E_2)/P(E_2)$ ).

We consider even more intersections of such events in the next section.

In [2] the value of  $\lambda$  for which the positional rule with weights  $(1, \lambda, 0)$  is most Condorcet efficient was determined. We call this ‘‘rules M’’ for brevity. The optimal value of  $\lambda$  is an algebraic irrational number given as the root of a polynomial of degree 8 and to 5 decimal places equals 0.37228. The corresponding value of the Condorcet efficiency is approximately 0.92546, only slightly more than that for Borda. To use this particular value of  $\lambda$  in computations similar to those above, it is probably best to switch to software that performs floating point computations in order to compute volumes. One such is **vinci**. We obtain for example that the joint Condorcet efficiency of the optimal rule and the Borda rule equals, to 5 decimal places, 0.89183.

plurality	rule M	Borda	antiplurality
0.0278 [16]	0.00131	0	0.0295 [16]

TABLE 2. Limiting probability of Borda’s paradox under IAC, 3 candidates

Rules	Elect same winner	Agree whole ranking
Anti-plurality and Borda	0.68717	0.56481
Anti-plurality and plurality (hence all scoring rules)	0.52315 [12]	0.29630
Plurality and Borda	0.82407	0.56481
All common rules	0.51268	

TABLE 3. Limiting probability of agreement of various rules under IAC, 3 candidates

*Borda’s Paradox.* We finish here by discussing **Borda’s Paradox**. Some rules can elect a Condorcet loser, namely a candidate that is beaten by every other when pairwise comparisons are made. The probability of this event for plurality and antiplurality has been studied under IAC in [16], and it has long been known to be zero for Borda. The methods in this section can be applied directly, since we need only replace the Condorcet winner conditions by the same ones with the direction of the inequality reversed. This shows that Borda’s Paradox occurs for plurality with probability  $1/36$ , agreeing with [16]. The corresponding results for Borda and antiplurality are 0 and  $17/576$ , corroborating the previous results. The probability that the most Condorcet efficient rule above elects the Condorcet loser is, as one might expect, very small. The results are shown in Table 2.

**3.3. When do all common rules elect the same winner?** For three-alternative elections, all positional voting rules elect the same winner in a given situation if and only if both plurality and antiplurality elect the same winner, since the vector of scores is a convex combination of those for the two extreme rules. The probability of this event has been investigated under IC but not under IAC as far as we are aware. In [23] Merlin, Tataru and Valognes also investigated the probability under IC that all positional rules and all Condorcet efficient rules yield the same winner (in this case, all scoring runoff rules also yield this same winner).

We again suppose that  $a$  is the winner. We want to compute the probability of the event  $P \cap A$  as described in the previous section. The relevant polytope has 18 vertices and  $m = 12$ . Its volume is  $113/77760$  and so the limiting probability that all positional rules yield the same winner for 3 alternatives under IAC is  $113/216$  (this confirms a result in [12]). We could also investigate the relationship between, say, plurality and Borda. They agree with probability  $89/108$ , whereas antiplurality and Borda agree with probability  $1039/1512$ .

The probability that all Condorcet rules and all positional rules elect the same winner given that the Condorcet winner exists is obtained easily via computation of  $\Pr(P \cap C \cap A)$  as above. The answer is  $3437/6480$ . The polytope involved has 29 vertices and  $m = 12$ .

We must also consider the case when no Condorcet winner exists. There are two cases corresponding to the two cycles  $a, b, c, a$  and  $a, c, b, a$ . In the first case, [23] shows that the rules all agree if and only if all positional rules give the ranking  $a, b, c$ , and this occurs if and only if both plurality and antiplurality give that ordering. The computation is straightforward as above and the probability of this event is only  $5/10368$ . The contribution from the cyclic case is therefore 32 times this, or,  $5/324$ , and the final result for the probability that all rules agree is  $10631/20736$ .

We can also consider the probability that two rules agree in their whole ranking, not just in the choice of winner. This is easily computed similarly to above: plurality and antiplurality agree on their whole ranking with probability  $8/27$ , while Borda and plurality agree with probability  $61/108$ . Borda and antiplurality also agree with probability  $61/108$ , which is clear by symmetry in any case.

**3.4. Abstention and Participation Paradoxes.** In [19] Lepelley and Merlin discuss various ways in which voters can attempt to manipulate an election by abstaining from voting. All scoring runoff

Underlying rule	PPP	NPP	PAP	NAP
plurality [19]	0	0.07292	0	0.04080
Borda	0.01389	0.02083	0.01042	0.01389
antiplurality [19]	0.03822	0	0.04253	0

TABLE 4. Limiting probability of participation paradoxes for scoring runoff rules under IAC, 3 candidates

rules and Condorcet rules suffer from this problem. Although abstaining turns out to be a dominated strategy for scoring runoff rules, it is still of interest to compute the probability that a situation may be manipulated in this way. Lepelley and Merlin carry this out under IC and IAC for scoring runoff rules based on plurality, antiplurality and Borda. For the latter (the Nanson rule) the limiting probability was not computed exactly (Table 5 of [19] refers to results of Monte Carlo simulation). We compute some exact values here.

We use the linear system given in [19]. Suppose that  $c$  is eliminated first and  $a$  then beats  $b$  in the runoff. The Positive Participation Paradox occurs when voters ranking  $a$  first are added to the electorate, and yet  $a$  then loses. This cannot happen when plurality is used at the first stage, but for other rules it can happen that  $b$  now loses the first stage, and  $a$  subsequently loses the runoff against  $c$ . Note that only voters with preference order  $acb$  can cause this to occur, and it can only occur when  $c$  originally beats  $a$  pairwise.

The system describing this set of voting situations contains the inequalities stating that  $a$  beats  $b$  and  $b$  beats  $c$  using the given scoring rule, and also that  $a$  beats  $b$  pairwise. In addition we have another constraint as described in [19] (note that  $n_6$  in the first equation on p.58 of that paper should be  $-n_6$ ). Carrying out the (by now routine) computation we obtain  $1/72$  which confirms the simulation result 0.14 referred to above. Note that the polytope involved has only 6 vertices and 6 facets but  $m = 18$ ; if  $e = 18$  (which we have not checked), it would be difficult to compute the Ehrhart polynomial using the old methods, which probably explains why only simulation results were obtained for the Nanson rule in the paper cited above.

Similarly we may compute the result for each of several other participation paradoxes. The results for the negative participation, positive abstention and negative abstention paradoxes (see [19] for definitions and characterizations of the polytopes) are respectively  $1/48, 1/96, 1/72$  confirming the earlier simulation results 0.020, 0.010, 0.14.

We can also perform the analogous computations for plurality and antiplurality runoff — the results confirm those in [19] and are shown in Table 4.

**3.5. The referendum paradox.** This gives an example where the variables describing our polytopes are slightly different.

In [7] the referendum or Compound Majority Paradox is studied. In the simplest case there are  $N$  equal sized districts each having  $n$  voters. There are two candidates  $a$  and  $b$  and voters in each district use majority rule to decide which candidate wins each district. The candidate winning a majority of districts is the winner of the election; the paradox occurs when this candidate would have lost if simple majority had been used in the union of all districts.

Among other things, the authors of [7] derive the probability of occurrence for  $N = 3, 4, 5$  under IAC using the older methods and state that they are not able to extend it to  $N \geq 6$ . Using the methods of the present paper it is easy to perform the computations for at least a few more values of  $N$ . Let  $n_i$  denote the number of voters voting for  $a$  in district  $i$ . The relevant set turns out to be

number of districts	3	4	5	6	7	9
probability	0.125 [7]	0.02083 [7]	0.15885 [7]	0.04063	0.20419	0.26954

TABLE 5. Limiting probability of referendum paradox under IAC, 3 candidates

described (ignoring ties for simplicity) by the union of polytopes of the form

$$\begin{aligned}
 n_i &\geq N/2 \text{ for } 1 \leq i \leq k && (a \text{ wins } k \text{ districts}) \\
 0 \leq n_i &\leq N/2 \text{ for } k+1 \leq i \leq N && (b \text{ wins } N-k \text{ districts}) \\
 \sum_i n_i &\leq Nn/2 && (b \text{ wins overall})
 \end{aligned}$$

for  $\lfloor N/2 \rfloor + 1 \leq k \leq N-1$ . The polytope  $P_n$  corresponding to  $k$  must have volume multiplied by  $2\binom{N}{k}$  to account for the symmetries of the problem. Note that there are  $(n+1)^N$  situations to consider and so the leading term in  $n$  of the Ehrhart polynomial gives the probability required.

Doing the analogous computation for  $N=3$  and  $N=4$  we obtain (as  $n \rightarrow \infty$ , in other words computing the volume of  $P_n/n$ ) results agreeing with [7]. However already for  $N=5$  we obtain  $61/384$  as opposed to their result  $55/384$ . For  $N=7$  we have  $9409/46080$ . The most complicated corresponding polytope in the last case has 36 vertices and 11 facets, whereas for  $N=9$  it has 91 vertices and 14 facets. We did not attempt to find the maximum value of  $N$  for which our software could obtain an answer; the answer was given essentially instantaneously for  $N=9$ . The conjecture in [7] that the probability tends to a limit of around 0.165 as  $N$  (odd) goes to infinity seems unlikely in the light of these results.

#### 4. SUMMARY AND DISCUSSION OF FUTURE WORK

We have shown that a wide variety of natural probabilistic questions for 3-alternative elections under IAC can be answered by applying standard algorithms for counting lattice points in, and computing volumes of, convex polytopes. For 4 or more alternatives the computations are conceptually the same but necessarily more complicated. However, the scope for extending results in the 3-candidate case to 4 or more candidates is obviously higher than for the older methods, which now appear to be completely superseded. One important point to notice is that many algorithms for volume computation have running times that are very sensitive to the number of defining hyperplanes and the number of vertices. Thus finding the most efficient description of the input system is important. It is certainly clear that further progress in this area will require researchers in social choice theory to understand in some detail how the fastest algorithms for lattice point counting and volume computation actually work. This may even lead to proofs for larger (or general) numbers of candidates when the polytopes concerned have a particularly nice structure.

Many questions naturally arise from our work here. One obvious line of attack is to try to find the optimal parameter for 3-alternative scoring rules that minimizes the probability of a certain undesirable behaviour occurring. The present authors are already engaged in carrying this out for the case of (naive, coalitional) manipulability. Numerical results obtained in [24] show that the answer may well be plurality, but this has never been proved. An attack on this problem along the lines of the approach in the present paper would require computation of volumes of a polytope whose defining constraints depend linearly in a parameter  $\lambda$ , and this requires considerable work as shown in [2]. Understanding of how to carry out such a computation would help in understanding the variation between positional rules. For example, the probability of electing a Condorcet loser is of order 0.03 for both plurality and antiplurality, but an order of magnitude smaller near the Borda rule, and as a function of  $\lambda$  is very flat there. Quantifying this type of variation analytically may show, for example, that it is not worth the trouble of replacing Borda by the Condorcet-optimal positional rule.

Another direction is to consider other probability models. For simplicity here we have not considered some common assumptions such as single-peaked preferences and the Maximal Culture Condition.

Many computations in these cases reduce to ones identical in spirit to those we have undertaken here. More general Pólya-Eggenberger distributions would lead to the more difficult issue of integrals of nonconstant probability densities over polytopes, but some results may be forthcoming there.

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