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OPTIMAL POLITICAL DISTRICTING

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Scope and Purpose—Given an area and its population units, we wish to divide the area into *m* districts such that each district has almost the same population of eligible voters (within a given tolerance), is contiguous, compact, and has a minimum number of split population units. This fair representation problem has been a great concern of the public for decades. The districting problem is also used in the design of sales territories. Re-districting occurs often because of population shifts or for political reasons. The purpose of this paper was to find a practical and automated operations research computer solution method for this problem.

Abstract—For the political districting problem, I propose the following solution methodology: (a) use Lagrangian relaxation to determine the centres of the districts, then, (b) use the transportation technique to assign population units to centres, and finally, (c) resolve the splitting problem by solving a sequence of capacitated transportation problems. This method is applied to the problem of determining the provincial districts for the City of Saskatoon, Canada, and the results are compared with the actual districting done in 1993 by the Electoral Boundaries Commission.

1. INTRODUCTION AND LITERATURE REVIEW

Given an area, e.g. a province or a city, and its population units, e.g. subdivisions or census tracts, we wish to divide the area into *m* districts such that each district has almost the same population of eligible voters (within a given tolerance, e.g. 5%). In addition, it is desirable that a district consists of contiguous population units and to be compact (close to square or circle). Frequently, to obtain population parity within the tolerance, population units have to be split between two or more districts. However, to reduce the time-consuming task of actually splitting population units and the ensuing confusion of voters, this splitting is to be minimized.

Political re-districting occurs frequently because of population shifts and because of political desires to change the total number of representatives. This problem is related to a problem in marketing called sales territory alignment. A computer method to solve political districting is desirable because (a) constructing districts with population parity within the tolerance and (b) keeping the number of split population units to a minimum are conflicting objectives which cannot be easily resolved manually.

The main literature on this problem started with Hess *et al.* [1] who modelled it as a location–allocation problem, but due to the computational difficulty solved it by the following heuristic: (a) start with an arbitrary set of centres of districts (a set of population units), (b) use the transportation technique to allocate population units to the centres, where the demand for each population unit is its population of eligible voters, the supply of each centre is the population quota for each district, and the distance between a centre and a population unit is the square of the Euclidean distance between their centres of gravity, (c) assign any split population units to the centre supplying the largest fraction of its demand, and (d) calculate the centre of gravity of each derived district and resolve the transportation problem with these centres until convergence of centres occurs.

Fleischmann and Paraschis [2] used a similar method, except for the following heuristic used to resolve the split population units in the solution of the transportation problem: (a) exclude the full and zero assignments, and identify the subtree F consisting of the split population units and their suppliers, (b) for each arc (i,j) in F, determine if supplier i, after adjustment to any other demands of k with (i,k) in F, needs to supply to j to remain within the tolerance, or if demand point j, after adjustment to any other suppliers k with (k,j) in F, needs to obtain supplies from i. If the answer is yes to either of these two questions, then a (partial) assignment on (i,j) is performed, and (c) if the answer to both questions in (b) for every arc (i,j) in F is negative, then a full assignment is made on (i,j) if the tolerance is not exceeded for supplier i.

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Thoreson and Liittschwager [3] used the following heuristic: (a) start with a reference population unit A, (b) find another population unit B furthest away from A, (c) build a district around B, including the closest population units to B until the district quota is just to be exceeded in which case include any contiguous population unit which will result in a population of eligible voters as close to the district quota as possible, (d) repeat with the furthest unassigned population unit B2 from A and build a district around B2, and (e) continue until all population units are assigned.

Garfinkel and Nemhauser [4] used the following exact method: (a) construct all sets of potential districts which are contiguous, compact, and have a total electorate within the tolerance, and (b) use a set partitioning integer linear programming model to minimize the maximum deviation of any chosen district from the district quota.

Among other heuristic methods which iteratively interchange population units between two districts are Bourjolly *et al.* [5] and Robertson [6]. Finally, Zoltners and Sinha [7] proposed a precedence network for population units to prevent discontiguity in districts. However, they assume that the centres of the districts are already fixed.

My approach is similar to [1] and [2], with the following differences: (a) I propose the use of Lagrangian relaxation to determine the centres of the districts, and (b) to resolve the splitting problem I propose the use of a sequence of capacitated transportation problems which at each stage tries to force one more shipment in F to zero. These methods are tested by using data for the City of Saskatoon, and the results are compared with the actual districting done in 1993 for the provincial legislature.

2. THE WAREHOUSE LOCATION MODEL

A **warehouse location model** for districting can be formulated as follows:

Let

N = number of population units

M = number of districts required

Q_j = number of eligible voters in population unit j (= demand of j)

Q = district quotient (= sum of Q_j M)

X_{ij} = proportion of demand of population unit j supplied by centre i (in population unit i). (N^2 variables)

c_{ij}' = square Euclidean distance from centre of population unit i to the centre of population unit j. This is similar to [1]. Other distance measures were also examined

$c_{ij} = c_{ij}' \times Q_j$

Y_i = indicator variable for population unit i:

= 1 if population unit i is to be chosen as the centre of a district

= 0 otherwise.

Minimize

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} X_{ij} \tag{1}$$

Subject to:

Demand:

$$\sum_{i=1}^{N} X_{ij} = 1 \quad \text{For each } j \tag{2}$$

Supply:

$$\sum_{j=1}^{N} Q_j X_{ij} = Q Y_i \quad \text{For each } i \tag{3}$$

$$\sum_{i=1}^{N} Y_i = M \tag{4}$$

$$X_{ij} \leq Y_i \quad \text{For each } j, \text{ for each } i \tag{5}$$

$$0 \leq X_{ij} \leq 1 \quad \text{For each pair } i, j \tag{6}$$

$$Y_i = 0 \text{ or } 1 \quad \text{For each } i. \tag{7}$$

Note that the population tolerance for each district is not used at this stage.

3. THE LAGRANGIAN RELAXATION METHOD

The procedure used here is a simplified version of that given in Beasley [8] who relaxes both the supply and the demand constraints.

First, normalize (3) by dividing it by Q. Let s_j and t_i be the multipliers associated with (2) and the normalized (3). The **Lagrangian lower bound programme (LLBP)** is:

Minimize

$$\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}X_{ij}+\sum_{j=1}^{N} s_j\left(1-\sum_{i=1}^{N} X_{ij}\right)+\sum_{i=1}^{N} t_i\left[-Y_i+\sum_{j=1}^{N}\left(\frac{Q_j}{Q}\right)X_{ij}\right]$$

Subject to: (4)–(7).

Defining

$$C_{ij}=c_{ij}-s_j+t_i\left(\frac{Q_j}{Q}\right)$$

LLBP becomes

Minimize

$$\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}X_{ij}-\sum_{i=1}^{N} t_iY_i+\sum_{j=1}^{N} s_j \tag{8}$$

Subject to: (4)–(7).

If $Y_i=0$, then the contribution of X_{ij} and Y_i to (8) is 0 (because X_{ij} will be set to 0 too), whereas if $Y_i=1$, this contribution is given by

$$a_i=-t_i+\sum_{j=1}^{N}\min(0,C_{ij})$$

because it is optimal to set $X_{ij}=1$ if $C_{ij}\leq>0$, and $X_{ij}=0$ if $C_{ij}>0$. Therefore, LLBP reduces to

Minimize

$$\sum_{i=1}^{N} a_iY_i+\sum_{j=1}^{N} s_j$$

Subject to:

$$\sum_{i=1}^{N} Y_i=M$$

$$Y_i=0 \text{ or } 1 \quad \text{For each } i.$$

This zero/one programme can be solved by inspection as follows: Sort the a_i in ascending order and set $Y_i=1$ for the first M population units. Set $X_{ij}=1$ if $Y_i=1$ and $C_{ij}\leq 0$, else set $X_{ij}=0$. Then, a valid lower bound on the optimal solution is given by

$$Z_{LB}=\sum_{i=1}^{N} a_iY_i+\sum_{j=1}^{N} s_j.$$

The near-optimal values for t_i, $i=1, \ldots, N$, and s_j, $j=1, \ldots, N$, are determined iteratively by the following **subgradient optimization method**:

Let

Z_{max} = maximum lower bound found

Z_{UB} = an upper bound for (1) (e.g. choose an arbitrary set of M centres, solve by the transportation technique, then set Z_{UB} = objective value)

n = number of subgradient iterations since Z_{max} last increased

f = step length parameter.

Step 1. Initialize $Z_{max}=-\infty$; $n=0$; $f=2$; $t_i=0$, $i=1, \ldots, N$; $s_j=c_{jj}$, $j=1, \ldots, N$.

Step 2. Solve LLBP with the current set of multipliers' and let the solution be Z_{LB}, (Y_i), and (X_{ij}). If

$Z_{LB} > Z_{max}$, set $n=0$, and $Z_{max}=Z_{LB}$. Else, set $n=n+1$.

Step 3. If $n=30$, then 30 iterations of the subgradient procedure have been performed without an increase in Z_{max}. Hence, halve the step length parameter f by setting $f=f/2$ and set $n=0$. Use the solution Z_{LB}, (Y_i) and (X_{ij}) corresponding to the best lower bound Z_{max} obtained so far.

Step 4. Calculate the subgradients G_j and H_i using

$$G_j = 1 - \sum_{i=1}^{N} X_{ij} \quad j=1, \ldots, N$$

and

$$H_i = -Y_i + \sum_{j=1}^{N} \left(\frac{Q_j}{Q}\right) X_{ij} \quad i=1, \ldots, N$$

Step 5. If $\sum_{j=1}^{N} G_j^2 + \sum_{i=1}^{N} H_i^2 = 0$ or $f < 0.005$, go to Step 7.

Step 6. Define the step size T by

$$T = \frac{(f)(Z_{UB} - Z_{LB})}{\sum_{j=1}^{N} G_j^2 + \sum_{i=1}^{N} H_i^2}$$

Update the Lagrange multipliers using

$$s_j = s_j + TG_j \quad j=1, \ldots, N$$

$$t_i = t_i + TH_i \quad i=1, \ldots, N$$

and go to Step 2.

Step 7. Solve the solution (Y_i) associated with the current maximum lower bound Z_{max} using a transportation technique. Terminate.

4. RESOLVING THE SPLITS

After determining the centres of districts by Lagrangian relaxation and solving the district composition problem by the transportation technique, the solution obtained will have at most $m-1$ population units which are split between two or more districts.

Lemma 1. The solution of the transportation technique will have at most $m-1$ split population units.

Proof. Because there are at most $m+N-1$ basic cells in this solution, at most $m-1$ demand points (population units) can have two or more cells in their associated columns, leaving at least $N-m+1$

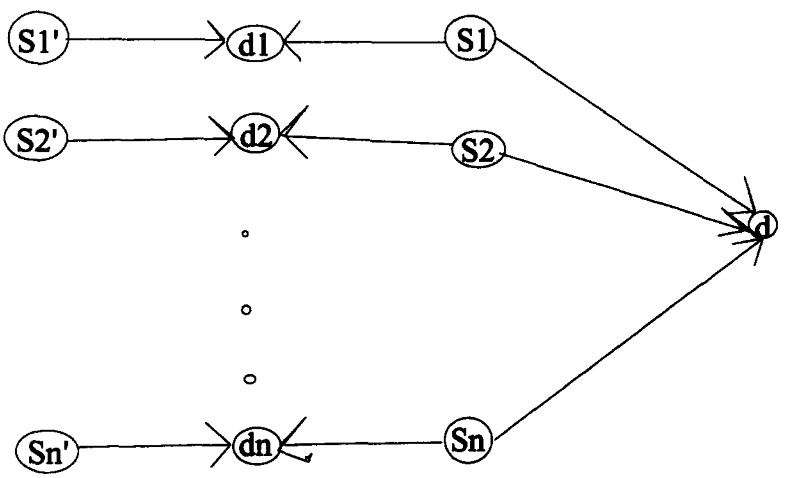

Fig. 1. The two-partition reduction to (SRP) used in the proof of Lemma 2.

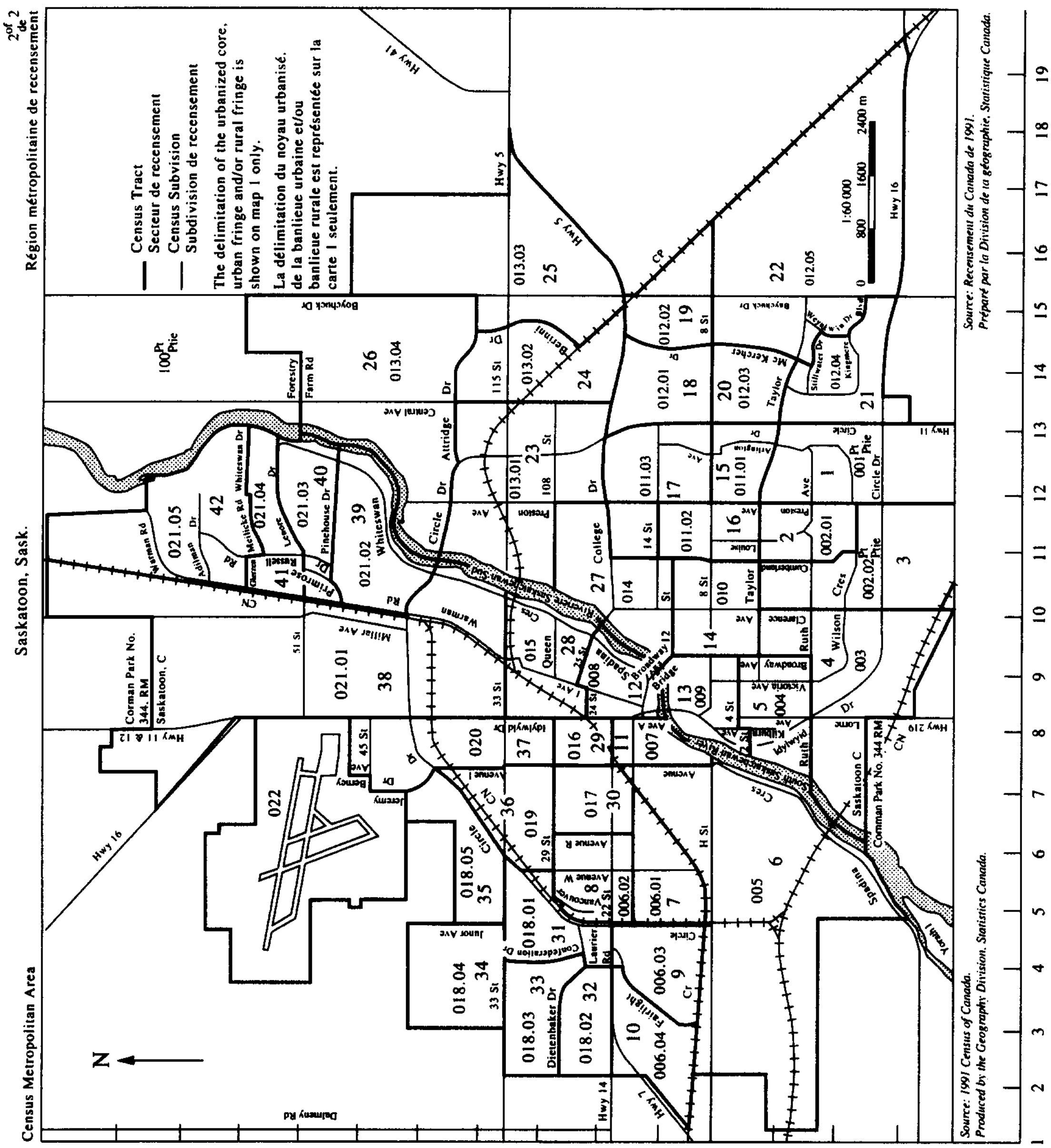

Fig. 2. Saskatoon 1991 Census Tracts.

Table 1. Population units (PU), number of eligible voters (1993), and the coordinates of centres of population units

PU	Voters	*X* coordinate	*Y* coordinate
1	4880	11.5	1.5
2	2350	10.5	2.5
3	3445	10	1
4	3455	8	1.5
5	4020	7.5	3
6	4550	4	3.5
7	3090	5	5
8	2990	5	6
9	3315	3	5
10	2665	2	5
11	1205	7	5.5
12	2330	8	5.5
13	3440	7.5	4.5
14	4585	9	4
15	2635	11.5	3.5
16	2365	10.5	4
17	2235	11.5	5
18	4050	13	5
19	3205	14.5	5
20	2605	13	3.5
21	5155	13	2
22	3670	15	2.5
23	3070	12	7.5
24	3835	13.5	7
25	1725	15	7.5
26	505	13	9
27	3475	9.5	5.5
28	3950	8.5	7
29	1255	7	6.5
30	2665	6	6
31	2615	4	7
32	2630	2	6
33	3825	2	7
34	2510	3	8.5
35	2150	5	8
36	3110	6	7.5
37	3480	7	8.5
38	3120	8	9
39	3920	10	10
40	3540	11	11
41	3310	10.5	11.5
42	3935	10.5	12.5

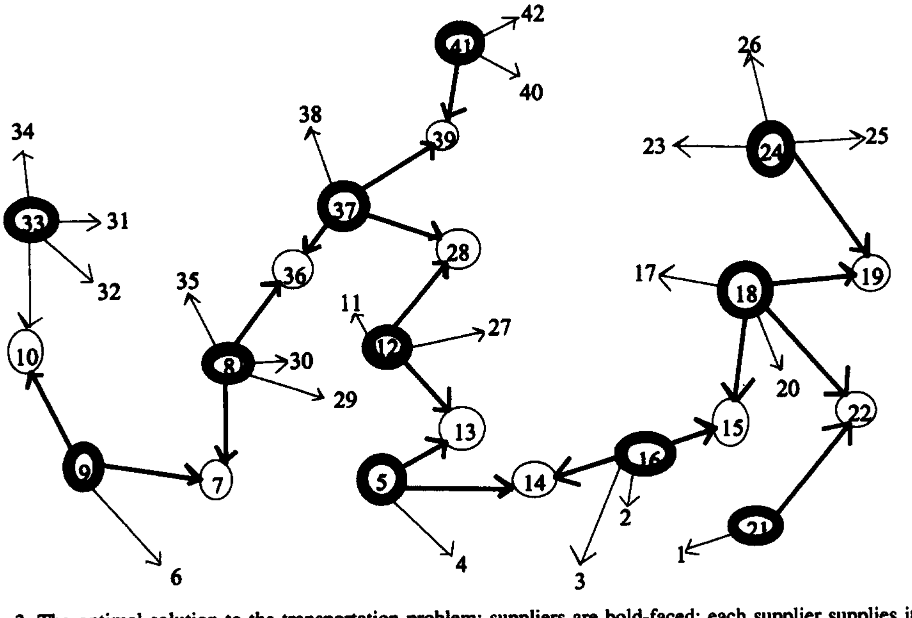

Fig. 3. The optimal solution to the transportation problem; suppliers are bold-faced; each supplier supplies its own population unit (not shown).

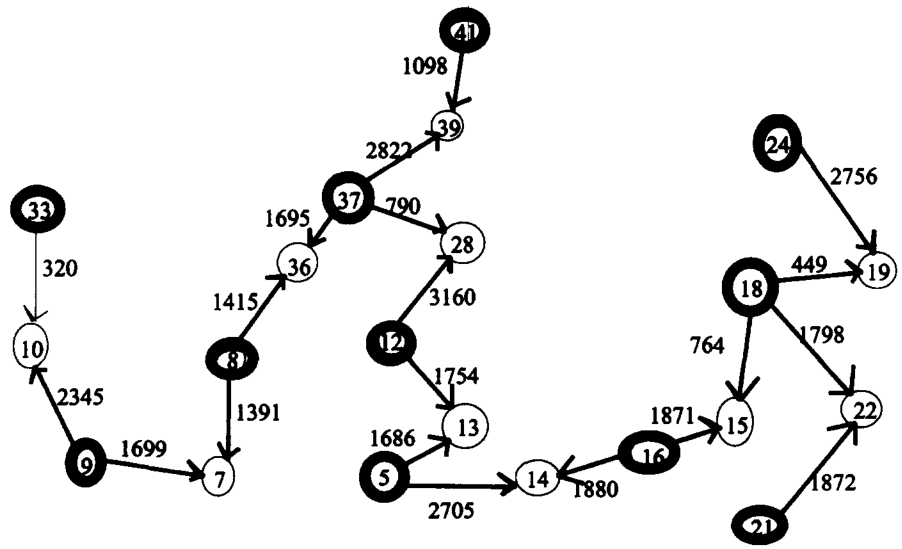

Fig. 4. The F subtree corresponding to the split population units in the solution of Fig. 3.

columns having exactly one basic cell. This is because $2(m-1)+(N-m+1)=N+m-1$. If a population unit is split among more than two districts, then the number of split population units will be less than $m-1$. ■

The split-resolution problem (SRP) can be formulated as follows:

Let

(x_{ij}) = solution of the transportation technique (in number of eligible voters not proportion)
J' = set of split population units
I' = set of centres adjacent to any split area
F = the graph with vertex set $I' \cup J'$ and the above adjacency [a forest since (x_{ij}) is a basic solution]
U_v = set of vertices adjacent to vertex v of F
$A_{\max} = Q(1+\text{tolerance})$, tolerance, e.g., $=0.05$
$A_{\min} = Q(1-\text{tolerance})$
a_i = size of district i without split population units.

Min. number of arcs (i,j) in F with $x_{ij}>0$

Subject to:

$$\sum_{i \in U_j} x_{ij} = Q_j \quad j \text{ in } J'$$

$$A_{\min} \le a_i + \sum_{j \in U_i} x_{ij} \le A_{\max} \quad i \text{ in } I' \qquad \text{(SRP)}$$

$$X_{ij} \ge 0 \quad i \text{ in } I', j \text{ in } J'.$$

Lemma 2. (SRP) is NP-hard.

Proof. We will reduce the NP-complete two-partition problem to (SRP). Given n real numbers $B_1, B_2, \ldots, B_n$, the two-partition problem asks if there is a set $S \subset \{1, \ldots, n\}$ such that

$$\sum_{j \in S} B_j = \sum_{j \notin S} B_j.$$

This problem is equivalent to a (SRP) with total number of suppliers in $F=2n$, supplier s_i having $A_{\max}-a_i=B_i$, and $A_{\min}-a_i=B_i$, $i=1, \ldots, n$; supplier s_i' having $A_{\max}-a_i=B_i$, and $A_{\min}-a_i=0$, $i=1, \ldots, n$; demand point d_j having $Q_j=B_j$, $j=1, \ldots, n$, and demand point d having demand

$$Q_d = \sum_{i=1}^{n} B_i/2$$

and the relationship as depicted in Fig. 1. ■

(SRP) can be formulated as a zero/one integer linear programme. Fleischmann and Paraschis [2] used the following heuristic to solve a problem which is equivalent to (SRP):

(a) For any arc (i,j) in F, an obligatory partial assignment $x_{ij}=\delta_{ij}$ takes place if

$$\delta_{ij}=\max(\delta_1,\delta_2)>0$$

where

$$\delta_1=Q_j-\sum_{h\in U_j,h\neq i}(A_{\max}-a_h)$$

$$\delta_2=A_{\min}-a_i-\sum_{k\in U_i,k\neq j}Q_k\,.$$

Then, a_i and Q_j are updated by $a_i=a_i+x_{ij}$ and $Q_j=Q_j-x_{ij}$, and another arc (k,l) is examined.

(b) If for every arc (i,j) in F, $\delta_{ij}\leq 0$, then an arbitrary full assignment can take place for any arc (i,j) if $a_i+Q_j\leq A_{\max}$. After this, repeat steps (a) (some δ_{ij} may become positive) and (b) until all arcs are assigned.

The application of Fleischmann and Paraschis heuristic may not reduce the number of arcs in F with

COST	d10	d7	d36	d39	d28	d13	d14	d15	
s33	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s33
s33e	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s33e
s9	.00000	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s9
s9e	.00000	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s9e
s8	9.0000	.00000	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	s8
s8e	9.0000	.00000	.00000	9.0000	9.0000	9.0000	9.0000	9.0000	s8e
s37	9.0000	9.0000	.00000	.00000	.00000	9.0000	9.0000	9.0000	s37
s37e	9.0000	9.0000	.00000	.00000	.00000	9.0000	9.0000	9.0000	s37e
s41	9.0000	9.0000	9.0000	.00000	9.0000	9.0000	9.0000	9.0000	s41
s41e	9.0000	9.0000	9.0000	.00000	9.0000	9.0000	9.0000	9.0000	s41e
s12	9.0000	9.0000	9.0000	9.0000	.00000	.00000	9.0000	9.0000	s12
s12e	9.0000	9.0000	9.0000	9.0000	.00000	.00000	9.0000	9.0000	s12e
s5	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	.00000	9.0000	s5
s5e	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	.00000	9.0000	s5e
s16	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	.00000	s16
s16e	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	.00000	s16e
s18	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	s18
s18e	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	.00000	s18e
s24	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s24
s24e	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s24e
s21	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s21
s21e	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000	s21e
DEMAND	2665.0	3090.0	3110.0	3920.0	3950.0	3440.0	4585.0	2635.0	DEMAND
COST	d10	d7	d36	d39	d28	d13	d14	d15	

Fig. 5. *Continued overleaf.*

COST	d19	d22	DD	SUPPLY	
s33	9.0000	9.0000	9.0000	.00000	s33
s33e	9.0000	9.0000	.00000	915.00	s33e
s9	9.0000	9.0000	9.0000	3449.0	s9
s9e	9.0000	9.0000	.00000	1190.0	s9e
s8	9.0000	9.0000	9.0000	2211.0	s8
s8e	9.0000	9.0000	.00000	1190.0	s8e
s37	9.0000	9.0000	9.0000	4712.0	s37
s37e	9.0000	9.0000	.00000	1190.0	s37e
s41	9.0000	9.0000	9.0000	503.00	s41
s41e	9.0000	9.0000	.00000	1190.0	s41e
s12	9.0000	9.0000	9.0000	4319.0	s12
s12e	9.0000	9.0000	.00000	1190.0	s12e
s5	9.0000	9.0000	9.0000	3796.0	s5
s5e	9.0000	9.0000	.00000	1190.0	s5e
s16	9.0000	9.0000	9.0000	3156.0	s16
s16e	9.0000	9.0000	.00000	1190.0	s16e
s18	.00000	.00000	9.0000	2416.0	s18
s18e	.00000	.00000	.00000	1190.0	s18e
s24	.00000	9.0000	9.0000	2161.0	s24
s24e	.00000	9.0000	.00000	1190.0	s24e
s21	9.0000	.00000	9.0000	1277.0	s21
s21e	9.0000	.00000	.00000	1190.0	s21e
DEMAND	3205.0	3670.0	6545.0		DEMAND
COST	d19	d22	DD	SUPPLY	

Fig. 5. The capacitated transportation model for F (arc capacities not shown).

$x_{ij}>0$, as will be seen in Section 6. Therefore, I propose the following heuristic for (SRP).

(a) Identify all arcs (i,j) in F with $\delta_{ij}\leq 0$, and call this set NA. These arc will always have a positive flow in them.
(b) Give all arcs in F an infinite capacity.
(c) Choose an arc (i,j) in F not in NA, set its capacity to zero, and solve a capacitated transportation problem as described below.
(d) If the problem obtained is feasible, go to next step; else reset the capacity of (i,j) back to infinity and continue.
(e) Choose another arc (k,l) in F not in NA, set its capacity to zero, and solve the capacitated transportation problem. Repeat as in (d) until all arcs in F not in NA are examined.

5. THE CAPACITATED TRANSPORTATION PROBLEM

The following formulation is similar to that given by Marlin [9], who used it to solve a (servicemen) districting problem with fixed district centers. Note that Marlin did not solve the split area problem. To represent the split resolution problem (which has a minimum and a maximum for each supplier) as a

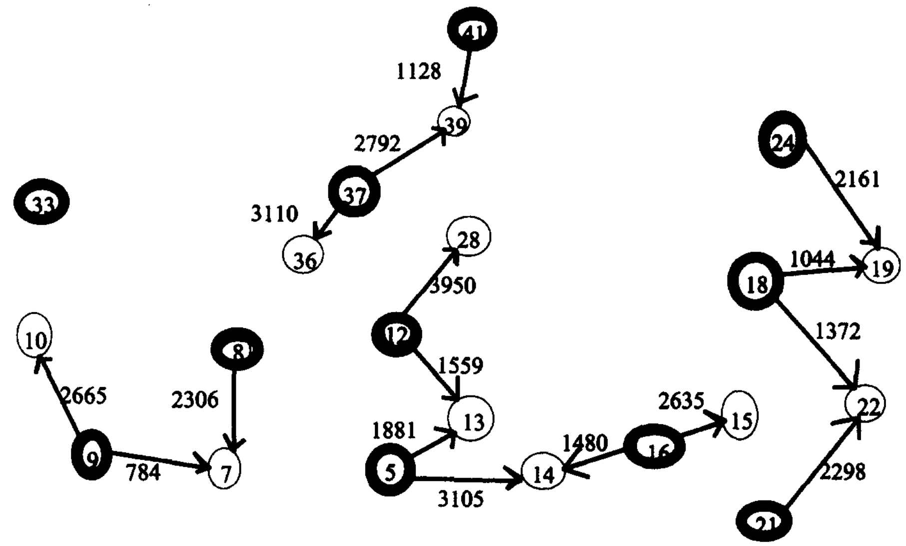

Fig. 6. The F subtree resulting from my heuristic.

transportation problem, we divide the shipments of a supplier i into two types: regular shipments x_{ij} which is part of the minimum supply $A_{\min} - a_i$, and the extra shipments x^e_{ij} (if needed, upto a total of $A_{\max} - A_{\min}$). Therefore, for each supplier s_i, we introduce a duplicate supplier s^e_i. Also, in addition to all demand points J', we introduce a dummy demand point DD to receive any extra shipments x^e_{ij} which is not shipped to any demand point in J'. Demand of DD is equal to (number of suppliers in F)$(A_{\max} - A_{\min})/2$.

Let

x_{ij} = shipment from s_i to d_j to satisfy (part of) $A_{\min} - a_i$

x^e_{ij} = shipment from s^e_i to d_j

$x^e_{i,DD}$ = shipment from s^e_i to DD

$c_{ij} = 9$ if (i,j) not in F (any positive cost will work)

$= 0$ if (i,j) in F

$c^e_{ij} = 9$ if (i,j) not in F

$= 0$ if (i,j) in F or $j =$ DD

Minimize $$\sum_{i\in I', j\in J'} c_{ij}x_{ij} + \sum_{i\in I', j\in J'} c^e_{ij}x^e_{ij} + \sum_{i\in I'} c^e_{i,DD}x^e_{i,DD}$$

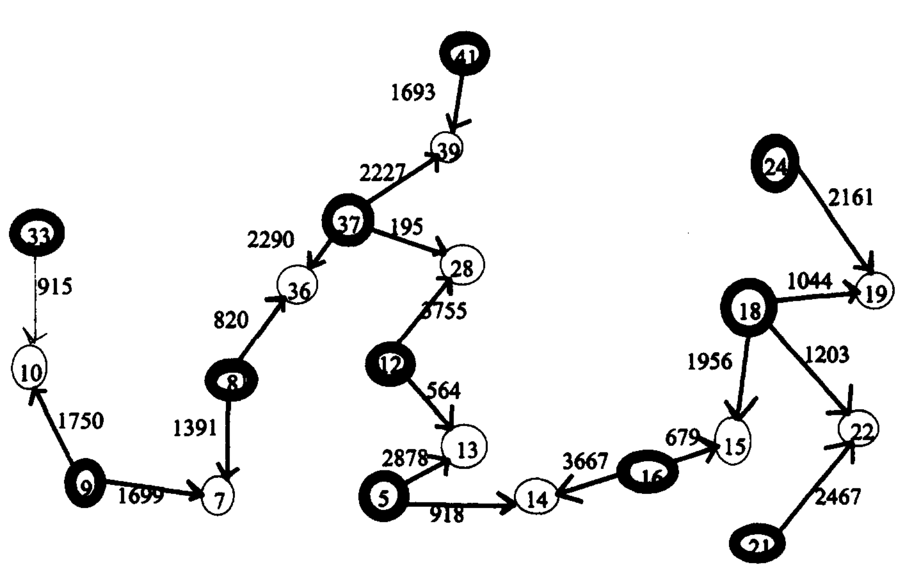

Fig. 7. The F subtree resulting from Fleischmann and Paraschis heuristic.

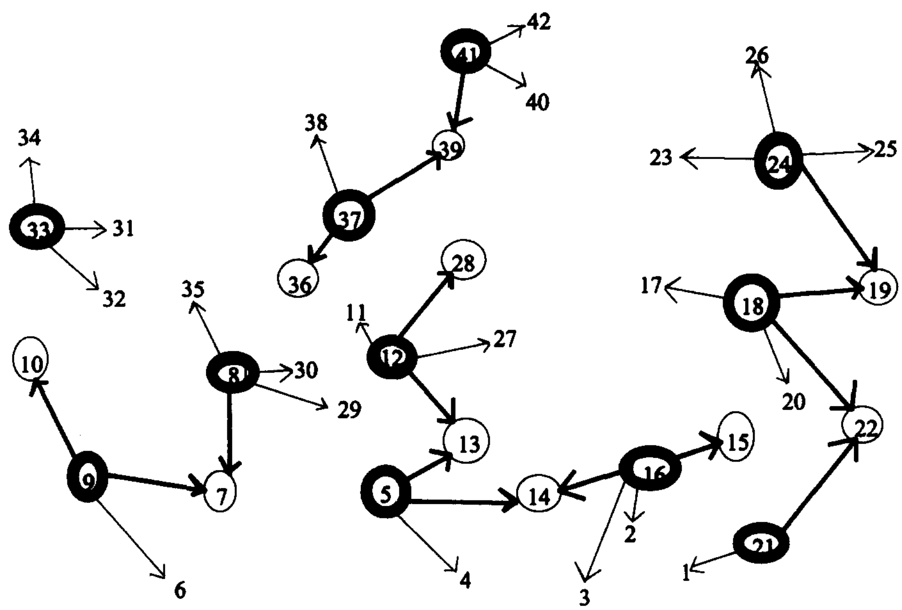

Fig. 8. The final solution (after split resolution heuristic).

Subject to:

$$\sum_{i\in I',j\in J'} x_{ij}+\sum_{i\in I',j\in J'} x^e_{ij}=Q_j \quad \text{For each } j \text{ in } J'$$

$$\sum_{i\in I'} X^e_{i,DD}=|I'|\times(A_{\max}-A_{\min})/2$$

$$\sum_{j\in J'} x_{ij}=A_{\min}-a_i \quad \text{For each } i \text{ in } I'$$

$$\sum_{j\in J'} x^e_{ij}+x^e_{i,DD}=A_{\max}-A_{\min} \quad \text{For each } i \text{ in } I'$$

Each cell's shipment is capacitated.

6. AN APPLICATION

The City of Saskatoon is to be partitioned into 11 provincial constituencies (districts). The population units were chosen to be census tracts, see Statistics Canada [10]. A census tract is generally a local community of population between 1000 to 5000 residents. The 42 census tracts were numbered population unit 1 to 42 as shown in Fig. 2.

The number of eligible voters in each census tract was obtained by summing all age groups 20 years or older residing in the tract from [10], and the city was grided horizontally and vertically to determine the coordinates of the centres of each census tract, see Table 1.

Then, the square Euclidean distance between all centres were computed, and all data were input to a computer programme which performed the Lagrangian relaxation method. The solution i.e. centres of districts, after 35 min on a 486 personal computer, in terms of population unit numbers is:

5, 8, 9, 12, 16, 18, 21, 24, 33, 37, and 41.

Then, a transportation software was used with the above centres as suppliers each having a capacity of $Q=130\,865/11=11\,896.8$, demands as given in Table 1, and square Euclidean distances. The optimal solution is displayed in Fig. 3.

Next, the split resolving method was used. The F subtree corresponding to split population units in the above solution and their split shipments are displayed in Fig. 4. The first capacitated transportation model to solve the split problem is displayed in Fig. 5. Note the tolerance was 5% or 595 voters.

Applying Fleischmann's δ_{ij} results in Step (a) identified the following arcs (the set NA):

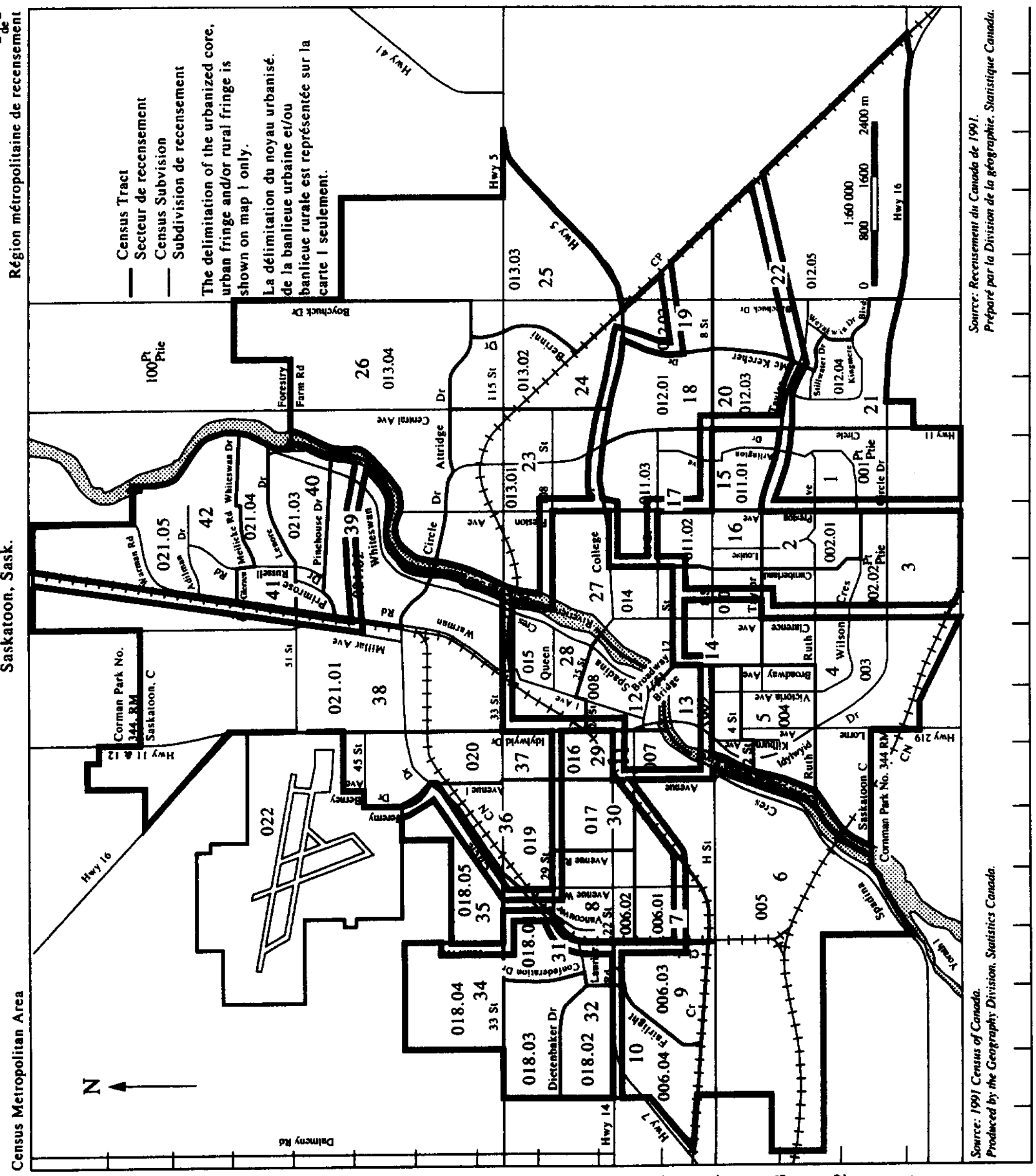

Fig. 9. Districts derived for Saskatoon in this paper.

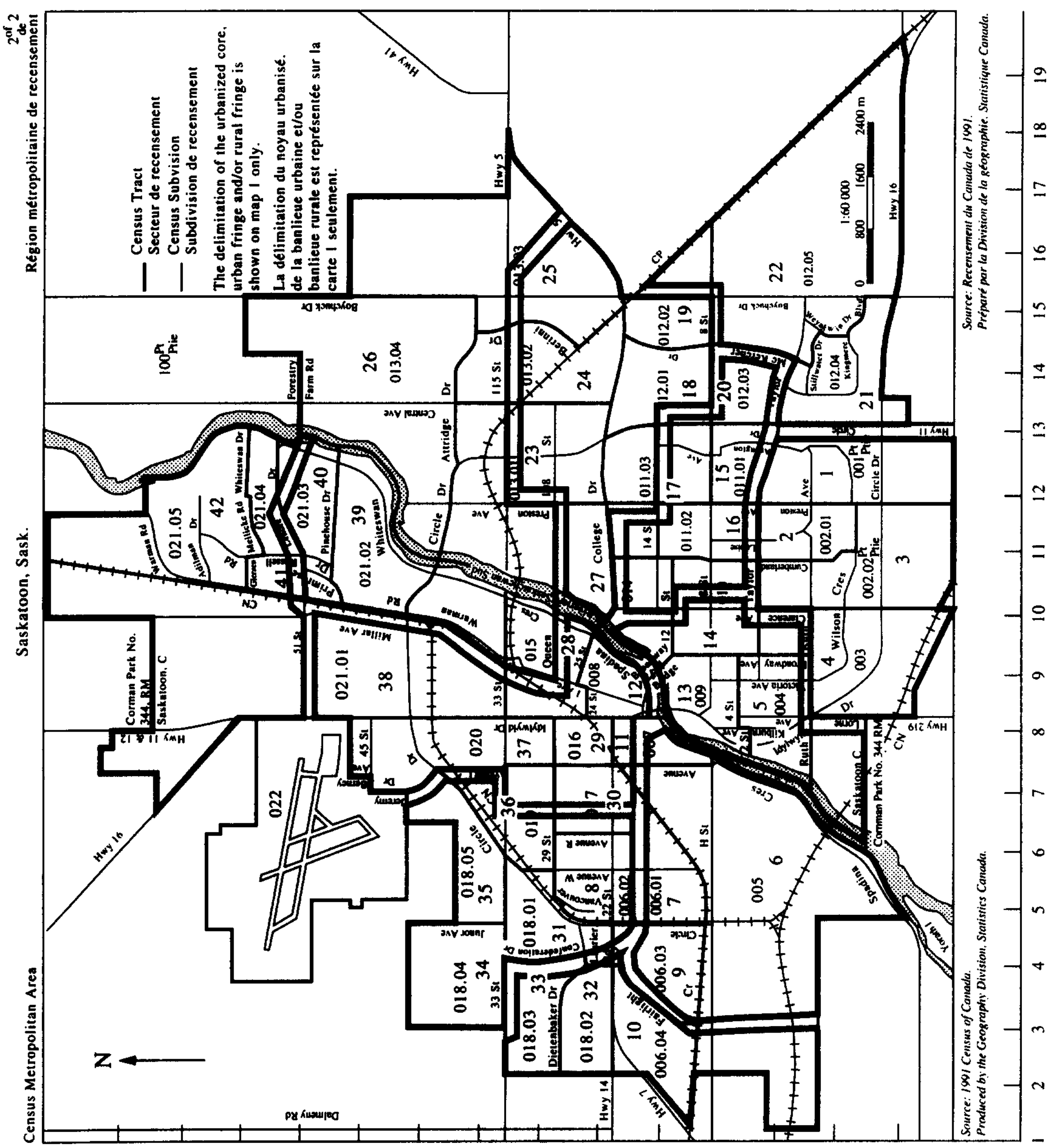

Fig. 10. Actual provincial constituencies of Saskatoon (1993).

(*s*9,*d*10), (*s*9,*d*7), (*s*37,*d*39), (*s*41,*d*39), (*s*12,*d*28), (*s*12,*d*13),
(*s*5,*d*14), (*s*16,*d*14), (*s*16,*d*15), (*s*18,*d*22), (*s*24,*d*19), (*s*21,*d*22).

Finally, the capacitated transportation model was repeatedly solved as follows (feasible has the objective value=0, infeasible has objective value>0):

Iteration	Capacity=0	Result
1	(*s*33,*d*10)	feasible
2	(*s*8,*d*7)	infeasible (reset capacity=∞)
3	(*s*36,*d*8)	feasible
4	(*s*37,*d*36)	infeasible (reset capacity=∞)
5	(*s*37,*d*28)	feasible
6	(*s*5,*d*13)	infeasible (reset capacity=∞)
7	(*s*18,*d*15)	feasible
8	(*s*18,*d*19)	infeasible (reset capacity=∞)

The *F* subtree has now changed to that displayed in Fig. 6. The number of split population units has been decreased by four, which happens to be the optimal solution to (SRP) in this case. In contrast, the Fleischmann and Paraschis heuristic results in a solution (see Fig. 7) with no elimination of arcs of *F*.

The final solution including the full assignments and those assignments given by my heuristic in Fig. 6 is displayed in Fig. 8. This solution is displayed in terms of city area in Fig. 9. The actual constituencies determined by the Electoral Boundaries Commission [11] are displayed in Fig. 10.

The following statistics compare the two solutions:

	My Solution	Actual Solution
Maximum deviation from quotient	5%	1%
Compactness measure	15	20
No. of discontiguous districts	1	0
No. of split areas	6	17

where compactness is measured as $\sum_{i=1}^{m} |L_i - W_i|$, L_i=maximum length of district i, W_i=maximum width of district i.

7. CONCLUSION

For the political districting problem I have proposed the following three-step solution methodology: (a) use Lagrangian relaxation to determine the centres of the districts, then, (b) use the transportation technique to assign population units to centres, and finally, (c) resolve the splitting problem by solving a sequence of capacitated transportation problems. The contributions of this paper are, (i) the use of Lagrangian relaxation to determine the centres of districts in the districting problem is new, and (ii), the use of a sequence of capacitated transportation problems to resolve the split area problem is also new.

As can be seen in Section 6, this three-step method is efficient and practical. It resulted in more compact districts and considerably less split areas (6) than the actual districts implemented (17). Compact areas are more convenient for the voters and the elected representatives in terms of travelling distances. A smaller number of split areas would mean reduced administrative costs in terms of record keeping and organisation of the district. Furthermore, our method needs a minimum number of manual intervention and is least affected by local politics.

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