ELSEVIER Mathematical Social Sciences 31 (1996) 133–146

mathematical social sciences

The likelihood of monotonicity paradoxes in run-off elections

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Received February 1995; revised September 1995

Abstract

A monotonicity paradox occurs when a voting system reacts in a perverse way to a change in individual opinions. The vulnerability of a voting system to monotonicity paradoxes is defined as the proportion of voting situations that can give rise to such paradoxes. In this paper we provide analytical representations of this vulnerability in the three-alternative case for two voting systems, i.e. plurality with run-off (f_1) and anti-plurality with run-off (f_2). Our results suggest that the vulnerability to monotonicity paradoxes is lower with f_1 than with f_2.

Keywords: Social choice; Voting theory; Run-off elections; Monotonicity paradoxes

1. Introduction

We consider a group of individuals or voters $N = \{1, 2, \dots, n\}$ who wish to collectively choose an alternative from a set A of m available alternatives (candidates in an election, competing projects, or allocations of goods between individuals). Each individual is assumed to have a linear preference ordering on A. Suppose that the $m!$ possible linear orders on A are numbered from 1 to $m!$ and let n_j be the number of individuals with the corresponding linear order. A *voting situation* (or simply a situation) is a vector of integers $x = (n_1, \dots, n_j, \dots, n_{m!})$ with $\sum_j n_j = n$, and the set of all possible voting situations is denoted by S. A *voting system* f is a mapping defined for every $x \in S$ that assigns a non-empty subset $f(x)$ of A to x.

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SSDI 0165-4896(95)00804-7

This definition suggests that the social choice process is *anonymous*, and in fact the two voting systems we consider in this paper are anonymous. Both systems are multistage procedures belonging to the class of *run-off point systems* (Richelson, 1980), or *scoring elimination methods* (Moulin, 1985). At each stage of the process the alternatives with the lowest 'scores', computed on the basis of a specific type of point system, are eliminated, and the process continues until no further elimination can take place. The remaining alternatives constitute the social choice set. The *plurality run-off system* (f_1) eliminates at each stage the alternatives with the fewest first-place votes, i.e. the score of an alternative is equal to the number of individuals who ranked it in first position. In contrast, under the *anti-plurality run-off system* (or Coombs' system), denoted f_2, the alternatives with the greatest number of last-place votes are eliminated; the score of an alternative is given by the number of individuals who did not rank it last.

Obviously, the systems f_1 and f_2 sequentially apply the plurality and anti-plurality systems respectively. One of the main reasons for introducing several stages in the social choice process is that run-off systems perform better than one-stage point systems with respect to Condorcet criteria. First, a Condorcet loser, i.e. an alternative that is beaten in all pairwise majority contests, cannot be elected under a run-off system, at least if tied elections are ignored. Secondly, the probability of electing the Condorcet winner, i.e. an alternative that beats everyone in majority comparisons, appears to be higher in run-off points systems than in one-stage points systems (see, for example, Gehrlein, 1982). Unfortunately, this advantage appears together with a serious flaw. We know from Smith (1973) that run-off point systems are not *monotonic*. The *monotonicity* principle is an important property in social choice theory. Roughly speaking, this principle requires that the reaction (or response) of the voting system to a change in individual preferences should not be perverse. More precisely, if a voting situation is altered so that the winning alternative gains more support, then this alternative must remain a winner. A violation of this general principle is called a *monotonicity paradox*. In what follows we will find it useful to distinguish between the two following monotonicity paradoxes.

Paradox ***M1*** (or the *more-is-less paradox*): the winner is ranked higher by one or more individuals (all else unchanged) and becomes a loser.

Paradox ***M2*** (or the *less-is-more paradox*): a loser is ranked lower by one or more voters (all else unchanged) and becomes a winner.

Numerous examples that illustrate paradox *M1* can be found in the literature (see, for example, Straffin, 1980; Fishburn and Brams, 1983). The following example illustrates paradox *M2* for both f_1 and f_2.

Table 1
Example voting situation

Preference order	n_i
1. *abc*	27
2. *acb*	5
3. *bac*	11
4. *bca*	27
5. *cab*	20
6. *cba*	10

Example 1. Suppose that $A = \{a, b, c\}$, $n = 100$, and consider the following voting situation x in Table 1. It is easy to check that $f_1(x) = f_2(x) = \{a\}$: c is eliminated in the first stage under f_1 as well as under f_2, and a beats b in the second stage. Consider the following modifications in individual opinions.

(i) Assume first that three individuals change their preference orders from *bca* to *cba*. Then b moves down and the situation becomes $x' = (27, 5, 11, 24, 20, 13)$. But now a gets the lowest number of first-place votes and, finally, $f_1(x') = \{b\}$.

(ii) Suppose next that two individuals change their rankings from *abc* to *acb*; then x becomes $x'' = (25, 7, 11, 27, 20, 10)$ and a obtains the highest number of last-place votes. Thus a is eliminated under f_2 and b wins the second round against c, i.e. $f_2(x'') = \{b\}$.

In both cases, a loser (b) gets less support and becomes a winner: we say that f_1 and f_2 are *vulnerable* to paradox *M2* in situation x.

From a theoretical point of view, violation of the monotonicity principle is certainly a serious drawback for a voting system, and many authors (e.g. Doron and Kronick, 1977) have argued against f_1 on the basis of its failure to satisfy monotonicity. However, what is the practical significance of monotonicity paradoxes? For a given voting system, is the occurrence of these paradoxes too rare to be of practical concern? Also, how do alternative voting systems compare with respect to their propensity to give rise to such paradoxes? The purpose of the present paper is to investigate these questions for f_1 and f_2 in the three-alternative case. The main results are stated in Section 2, and some implications thereof are discussed in Section 3. Concluding remarks are given in Section 4.

2. Main results

Let $A = \{a, b, c\}$. Given three alternatives and assuming anonymous voters, the total number of distinguishable voting situations depends on the number of individuals in the following way (see, for example, Gehrlein and Fishburn, 1976):

$$|S| = \binom{n+5}{5} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120}. \tag{1}$$

Given a monotonicity paradox M and a voting system f, we define the *vulnerability* of f to M as the proportion of voting situations in which f is vulnerable to M, i.e. the proportion of situations that can give rise to M under f. Note that the underlying probabilistic assumption of this approach is the so-called 'impartial anonymous culture' condition (see Berg and Lepelley, 1994, for comments on this condition).

Clearly, a voting system that is vulnerable to $M1$ (respectively $M2$) is also vulnerable to $M2$ ($M1$), but this does not imply that the number of situations giving rise to $M1$ is equal to the number of situations giving rise to $M2$: the vulnerability to $M1$ a priori is different from the vulnerability to $M2$. For a given number n of individuals, we denote by $V_{ij}(n)$ the vulnerability of f_i to Mj, $i,j \in \{1,2\}$. Hence $V_{ij}(n) = |X_{ij}|/|S|$, where X_{ij} is the set of situations for which f_i is vulnerable to Mj.

To simplify our calculations, in this paper we ignore the problem of tied elections: we assume that one and only one alternative is eliminated in the first stage as well as in the second. It is clear that this assumption alters the results only for small values of n.

2.1. Vulnerability to paradox M1

The first proposition, due to Berg and Lepelley (1993), provides a characterization of those situations in which the plurality run-off system is vulnerable to paradox $M1$ in three-alternative elections. In this proposition and in the remaining sections of the paper the six possible rankings on the three alternatives are numbered as in Example 1, and we write n_{ij} for $n_i + n_j$.

Proposition 1 (Berg and Lepelley, 1993). *Under f_1, a situation $x = (n_1, \ldots, n_6)$, such that $f_1(x) = \{a\}$, can give rise to paradox M1 if and only if*

$$(n_{34} + n_6 > n/2 \text{ and } n_{34} > n/4) \quad \text{or} \quad (n_{56} + n_4 > n/2 \text{ and } n_{56} > n/4).$$

From Proposition 1 it is possible to derive an explicit formula for $V_{11}(n)$. For computational expediency, the result is given modulo 12.

Corollary 1.

$$V_{11}(n) = \frac{n(26n^4 - 325n^3 - 760n^2 + 9360n + 19584)}{576(n+1)(n+2)(n+3)(n+4)(n+5)}$$

for $n \in \{24, 36, 48, 60, \ldots\}$

and $\lim_{n\to\infty} V_{11}(n) = 13/288 \cong 0.0451$.

Proof. Let $X^a_{11}(X^b_{11}, X^c_{11})$ be the subset of X_{11} such that $a(b,c)$ is the chosen alternative. By the symmetry of a, b and c in S, $|X^a_{11}| = |X^b_{11}| = |X^c_{11}|$. Moreover, a

is the chosen alternative if and only if $[(n_{34} < n_{12}$ and $n_{34} < n_{56}$ and $n_{12} + n_3 > n/2)$, or $(n_{56} < n_{12}$ and $n_{56} < n_{34}$ and $n_{12} + n_5 > n/2)]$, i.e. either b is eliminated in the first round and a beats c in the second round, or c is eliminated in the first stage and a beats b in the second (recall that we ignore tied elections). Hence it follows from Proposition 1 that a situation x belongs to X^a_{11} if and only if

$$(n_{34} < n_{12},\ n_{34} < n_{56},\ n_{12} + n_3 > n/2,\ n_{34} + n_6 > n/2 \text{ and } n_{34} > n/4)\,, \tag{2}$$

or

$$(n_{56} < n_{12},\ n_{56} < n_{34},\ n_{12} + n_5 > n/2,\ n_{56} + n_4 > n/2 \text{ and } n_{56} > n/4)\,. \tag{3}$$

Let (Y', Y'') be a partition of X^a_{11} such that $x \in Y'$ if and only if (2) is true and $x \in Y'$ if and only if (3) is true. By symmetry, $|Y'| = |Y''|$. Suppose now that n is a multiple of three and four, then (2) and $n_{12} + n_{34} + n_5 + n_6 = n$ imply that for all situations in Y' it must be true that

$$n/4 + 1 \leq n_{34} \leq n/3 - 1\,, \quad n_{34} + 1 \leq n_{12} \leq n - 2n_{34} - 1\,, \quad 0 \leq n_1 \leq n_{12}\,,$$
$$n/2 - n_{34} + 1 \leq n_6 \leq n - n_{12} - n_{34} \quad \text{and} \quad n/2 - n_{12} + 1 \leq n_3 \leq n_{34}\,.$$

Here the cardinality of Y' is given by evaluating the number of combinations of the n_i's that satisfy the above inequalities. Using summation formulas for powers of integers, we obtain

$$|Y'| = \frac{n(26n^4 - 325n^3 - 760n^2 + 9360n + 19585)}{414720}\,. \tag{4}$$

Since $|X_{11}| = |X_{11}| + |X^b_{11}| + |X^c_{11}| = 3|X^a_{11}| = 3(|Y'| + |Y''|) = 6|Y'|$ and $V_{11}(n) = |X_{11}|/|S|$, the desired result is deduced from (1) and (4). Q.E.D.

Note that the above formula has been checked by complete enumeration for fairly small values of n, and this remark applies to all similar relations given in this paper.

We now consider the vulnerability of f_2 to paradox *M1*. The following proposition makes possible the computation of $V_{21}(n)$.

Proposition 2. *Under f_2, a situation $x = (n_1, \ldots, n_6)$ such that $f_2(x) = \{a\}$ can give rise to paradox M1 if and only if*

$$(n_{46} + n_3 > n/2 \text{ and } n_{25} < n_{13} + n_4) \quad \text{or} \quad (n_{46} + n_5 > n/2 \text{ and } n_{13} < n_{25} + n_6)\,.$$

Proof. Consider a situation x such that $f_2(x) = \{a\}$. We denote by S_x the set of situations obtained from x by moving up alternative a in at least one individual order (all others unchanged); an element of S_x is denoted by $x' = (n'_1, \ldots, n'_6)$. Observe that the definition of x', together with $f_2(x) = \{a\}$, imply that a cannot be eliminated in the first stage under f_2 in x'.

To prove the necessity part of Proposition 2, we have to show that if one of the following statements holds

(α) ($n_{46}+n_3<n/2$ and $n_{46}+n_5<n/2$),

(β) ($n_{46}+n_3>n/2$ and $n_{25}>n_{13}+n_4$),

(γ) ($n_{46}+n_5>n/2$ and $n_{13}>n_{25}+n_6$),

then x cannot give rise to $M1$, i.e. $f_2(x')=\{a\}$ for any $x'\in S_x$. Suppose first that (α) holds. In this case, a is both the f_2 winner and the Condorcet winner in x, and it is easily checked that this remains true in x'; hence $f_2(x')=\{a\}$ for any $x'\in S_x$. Suppose now that (β) holds; the first inequality in (β) means that more than one-half of the individuals prefer b to a in x. Since $f_2(x)=\{a\}$, a majority of individuals prefer a to c in x, and this is also true in x'. Hence, $f_2(x')\neq\{c\}$ for any $x'\in S_x$. Suppose that $f_2(x')=\{b\}$ for some x' in S_x; this implies that the number of last positions of b is smaller than the number of last positions of c in x', i.e. $n'_{25}<n'_{13}$. However, the definition of x' implies $n'_{25}\geq n_{25}$ and $n'_{13}+n'_4=n_{13}+n_4$ (since the majority relation between b and c is unchanged). From these relations we deduce that $n_{13}+n_4>n_{25}$, which contradicts (β). Hence, b cannot be the winner in x' and we conclude that $f_2(x')=\{a\}$ for every $x'\in S_x$. Replacing b by c in the above analysis, we obtain a similar conclusion for the case where (γ) holds.

To prove the sufficiency part of Proposition 2, we assume first that ($n_{46}+n_3>n/2$ and $n_{25}<n_{13}+n_4$). Consider a situation $x'=(n'_1,\ldots,n'_6)$ defined in the following way: $n'_1=n_1$, $n'_2=n_2$, $n'_3=n_3+n'_4=0$, $n'_5=n_5$ and $n'_6=n_6$. Observe that $n_4>0$ (if $n_4=0$, then $n_{25}<n_{13}$ and, as a result, c and not b would have been removed in the first round, which would have led to b and not a being the winner in the second round). Thus, from x to x', some individuals (at least one) change their preference orders from bca to bac. Hence x' belongs to S_x. Moreover, we have $n'_{25}=n_{25}$ and $n'_{13}=n_{13}+n_4$. Since $n_{25}<n_{13}+n_4$, we obtain $n'_{25}<n'_{13}$, and from this inequality the result is that c is eliminated in the first stage under f_2 in x' (recall that a cannot be eliminated in the first stage in x'). Since $n'_3+n'_4+n'_6=n_3+n_4+n_6>n/2$ by hypothesis, we finally obtain $f_2(x')=\{b\}$, i.e. paradox $M1$ occurs. Let us assume now that ($n_{46}+n_5>n/2$ and $n_{13}<n_{25}+n_6$) and consider a situation x' defined as follows: $n'_1=n_1$, $n'_2=n_2$, $n'_3=n_3$, $n'_4=n_4$, $n'_5=n_5+n_6$ and $n'_6=0$. We then obtain $f_2(x')=\{c\}$, and this completes the proof. Q.E.D.

Corollary 2.

$$V_{21}(n)=\frac{n(2n^4-15n^3+60n^2-180n-432)}{36(n+1)(n+2)(n+3)(n+4)(n+5)}\quad \textit{for } n\in\{24,36,48,60,\ldots\}$$

and $\lim_{n\to\infty}V_{21}(n)=1/18\cong 0.0556$.

The proof of this result is easy and very similar to the proof of Corollary 1.

2.2. Vulnerability to paradox M2

We now turn to the study of paradox *M2*, or the less-is-more paradox. We begin by characterizing the situations for which the plurality run-off system f_1 is vulnerable to *M2*.

Proposition 3. *Under f_1, a situation $x = (n_1, \ldots, n_6)$ such that $f_1(x) = \{a\}$ can give rise to paradox M2 if and only if*

$$(n_{12} < n/3)$$

and

$$[(n_{34} > n_{12},\ n_4 > n_{12} - n_{56} \text{ and } n_{12} + n_2 < n/2) \text{ or } (n_{56} > n_{12},\ n_6 > n_{12} - n_{34} \text{ and } n_{12} + n_1 < n/2)]\,.$$

Proof. See Appendix A.

Corollary 3.

$$V_{12}(n) = \frac{n(17n^4 - 495n^3 + 4200n^2 - 6480n - 24192)}{864(n+1)(n+2)(n+3)(n+4)(n+5)}$$

for $n \in \{24, 36, 48, 60, \ldots\}$

and $\lim_{n\to\infty} V_{12}(n) = 17/864 \cong 0.0197$.

The proof follows closely the proof of Corollary 1.

The final proposition characterizes those situations for which f_2 is vulnerable to *M2*.

Proposition 4. *Under f_2, a situation $x = (n_1, \ldots, n_6)$ such that $f_2(x) = \{a\}$ can give rise to paradox M2 if and only if*

$$n_{46} > n/3$$

and

$$[(n_{46} > n_{25} \text{ and } n_{46} + n_4 > n/2) \text{ or } (n_{46} > n_{13} \text{ and } n_{46} + n_6 > n/2)]\,.$$

Proof. See Appendix A.

Corollary 4.

$$V_{22}(n) = \frac{n(7n^4 + 30n^3 - 600n^2 + 6048)}{108(n+1)(n+2)(n+3)(n+4)(n+5)} \quad \text{for } n \in \{24, 36, 48, 60, \ldots\}$$

and $\lim_{n\to\infty} V_{22}(n) = 7/108 \cong 0.0648$.

Table 2
Vulnerability of f_1 and f_2 to monotonicity paradoxes

n	Paradox *M1* $V_{11}(n)$	Paradox *M1* $V_{21}(n)$	*Paradox M2* $V_{12}(n)$	*Paradox M2* $V_{22}(n)$
24	0.01152	0.02274	0.00202	0.03739
36	0.01936	0.03036	0.00493	0.04591
48	0.02439	0.03518	0.00723	0.05046
60	0.02781	0.03848	0.00898	0.05237
72	0.03027	0.04087	0.01031	0.05516
84	0.03213	0.04267	0.01136	0.05654
96	0.03357	0.04408	0.01220	0.05757
108	0.03473	0.04522	0.01289	0.05837
120	0.03568	0.04615	0.01347	0.05901
132	0.03647	0.04693	0.01396	0.05954
144	0.03714	0.04759	0.01437	0.05998
156	0.03771	0.04815	0.01473	0.06035
168	0.03821	0.04864	0.01505	0.06067
180	0.03864	0.04907	0.01532	0.06094
192	0.03903	0.04945	0.01557	0.06119
⋮				
Limit	0.04514	0.05556	0.01968	0.06481

Note: f_1, plurality run-off system; f_2, anti-plurality run-off system.

Proof. Omitted.

Table 2 gives the $V_{ij}(n)$ values for each $i \in \{1,2\}$, $j \in \{1,2\}$ and $n \in \{24, 34, 48, \ldots, 192\}$. These figures show that the proportion of situations that can give rise to monotonicity paradoxes increases with the number of voters, ranging from 0.2% to 4.5% for the plurality run-off system f_1 (according to the number of individuals and the paradox), and from 2% to 6.5% for the anti-plurality run-off system f_2. Hence, f_1 appears to be less vulnerable than f_2, especially when we consider paradox *M2*. Comparing f_1 and f_2, we find that f_2's vulnerability to *M2* is greater by a factor of 3.2 when the electorate is large and by a factor of 18.5 when the election involves 24 voters.

3. Extensions

The analysis we have presented in the preceding section can be extended in at least three directions.

3.1. Global vulnerability

Our results allow us to calculate a *global* measures for the vulnerability of f_1 and f_2 to monotonicity paradoxes, i.e. a measure that takes into account both *M1* and $M2$. Let $\tilde{V}_i(n)$ be the global vulnerability of f_i, $i = 1, 2$. In what follows we compute the limiting values $\tilde{V}_1(\infty)$ and $\tilde{V}_2(\infty)$. Let $p_j = n_j/n$ be the proportion of individuals with the preference order j: a situation is now a vector $p = (p_1, \ldots, p_6)$ with $p_j \geq 0$ and $\Sigma\, p_j = 1$. From Propositions 1 and 3 it is easy to see that there exist situations in which both paradoxes $M1$ and $M2$ can occur under f_1. Moreover, these situations are characterized by the following inequalities when the winner is a (we use the notation $p_{ij} = p_i + p_j$):

$$(p_{34} < p_{12},\ p_{12} < p_{56},\ p_{12} + p_3 > 1/2,\ p_{34} + p_6 > 1/2,\ p_{34} > 1/4,\ p_{12} < 1/3,\\ p_6 > p_{12} - p_{34} \text{ and } p_{12} + p_1 < 1/2)\,, \qquad (5)$$

or

$$(p_{56} < p_{12},\ p_{12} < p_{34},\ p_{12} + p_5 > 1/2,\ p_{56} + p_4 > 1/2,\ p_{56} > 1/4,\ p_{12} < 1/3,\\ p_4 > p_{12} - p_{56} \text{ and } p_{12} + p_2 < 1/2)\,. \qquad (6)$$

Since $\Sigma\, p_j = 1$, the set of inequalities (5) is equivalent to

$$1/4 < p_{34} < 1/3,\ p_{34} < p_{12} < 1/3,\ 1/2 - p_{12} < p_3 < p_{34},\\ 1/2 - p_{34} < p_6 < 1 - p_{34} - p_{12} \text{ and } 0 < p_1 < 1/2 - p_{12}\,. \qquad (7)$$

When n tends to infinity, the proportion of situations that satisfy these inequalities can be computed by evaluating the following multiple integral over the domain defined by (7):

$$\int\!\!\int\!\!\int\!\!\int\!\!\int 120\, \mathrm{d}p_{34}\, \mathrm{d}p_{12}\, \mathrm{d}p_3\, \mathrm{d}p_6\, \mathrm{d}p_1\,.$$

We obtain 17/13824. By observing that

(i) the proportion of situations that verify (5) is equal to the proportion of situations that verify (6), and

(ii) a, b and c are symmetric in the set of all possible situations,

we conclude that the proportion of situations that can give rise to both $M1$ and $M2$ under f_1 is given by $(2 \times 3 \times 17)/13824 = 17/2304$. Hence, from the corollaries following Propositions 1 and 3, we obtain

$$\tilde{V}_1(\infty) = 13/288 + 17/864 - 17/2304 = 397/6912 \cong 0.0574\,.$$

We now consider the global vulnerability of f_2. Starting from Propositions 2 and 4 and using a similar approach as above, we obtain

$$\tilde{V}_2(\infty) = 1/18 + 7/108 - 5/1296 = 151/1296 \cong 0.1165\,.$$

Hence, for large electorates, the global vulnerability of f_2 to monotonicity paradoxes is twice as high as the global vulnerability of f_1.

3.2. Monotonicity paradoxes and strategic manipulation

It is important to emphasize that the results given here evaluate the proportion of situations that are *potentially* paradoxical: paradoxes occur if and only if some individual preferences are modified in a specific way. However, are such modifications likely to occur? In other words, can we find rational arguments that justify these modifications? If the answer is negative, then monotonicity paradoxes remain nominal and their practical relevance is limited. We show in this subsection that such rational (strategic) arguments can, indeed, be found in some cases, but not in every case.

Consider Example 1(i) and assume that preferences are sincere in situation x. The three individuals who change their preference orders prefer b to a. Since this modification makes b the winner, they have a good reason for doing so. In this case the occurrence of paradox *M2* can be expected. On the other hand, the two individuals who change their preferences in Example 1(ii) prefer a (the winner in x) to b (the winner in x''). In such a case it is difficult to find a convincing argument that justifies a change in preference orders,

One can easily prove (see Appendix B) the following assertion for three-alternative elections: *under f_1 (respectively f_2), every situation that can give rise to M2 (M1) involves a modification in individual preferences that can be justified by strategic arguments*. A similar conclusion holds neither for f_2 and *M2* (as shown by Example 1(ii)), nor for f_1 and *M1*, i.e. strategic arguments can be found in some cases, but not in every case.

Thus, if individuals are rational and perfectly informed, the proportion of situations in which monotonicity paradoxes are likely to occur in three-alternative elections with large electorates is at least 1.97% for f_1 and at least 5.56% for f_2. This result suggests that manipulation possibilities are more frequent under f_2 than under f_1. This is in accordance with the conclusions of Lepelley and Mbih (1994), who compare the vulnerability of these two voting systems to strategic manipulation by coalitions of individuals.

3.3. Single-peaked preferences

Our calculations assume that every voting situation is equally likely to occur. In some political or economic contexts such an assumption is questionable, in view of the fact that some preference rankings appear to be very unlikely. One common way to take this into account is to assume that preferences are single-peaked. When preferences are single-peaked and three alternatives are in contention, every voter agrees to consider that (at least) one of these alternatives is not the worst. Without loss of generality, we assume that this alternative is b. Hence, in our framework, the single-peakedness assumption implies $n_2 = n_5 = 0$.

Table 3
Vulnerability of f_1 and f_2 to monotonicity paradoxes with single-peaked preferences and large electorates ($n \to \infty$)

	Paradox *M1*	Paradox *M2*
f_1	0.0174	0
f_2	0	0.0463

Using this observation and Propositions 2 and 3, it can be shown (see Appendix C) that with single-peaked preferences

- *M2 never occurs under* f_1;
- *M1 never occurs under* f_2.

Now, let us assume that every single-peaked situation is equally likely to occur. Under this assumption, and following an approach developed by Lepelley (1993), it is easy with the help of Propositions 1 and 4 to calculate the vulnerability of f_1 to paradox *M1* and the vulnerability of f_2 to paradox *M2*. For large electorates ($n \to \infty$), we obtain 5/288 and 5/108, respectively. Table 3 summarizes the results.

It is clear from Table 3 that the single-peakedness assumption significantly reduces the vulnerability of both f_1 and f_2; and it turns out that f_1 performs better than f_2, also when preferences are single-peaked.

4. Conclusion

This paper investigates the likelihood of monotonicity paradoxes in run-off elections. Although the results given here are limited to the three-alternative cases, they are sufficient to suggest two main conclusions. First, it seems difficult to claim that monotonicity paradoxes are extremely rare and have no practical relevance (at least when the electorate is large). Secondly, the plurality run-off system appears to be less vulnerable to these paradoxes than the anti-plurality run-off system, and this conclusion proves an argument for choosing the former system rather than the latter.

Acknowledgements

Financial support for this research was provided by the Swedish Council for Research in the Humanities and Social Sciences and (Lepelley) the Swedish Institute.

Appendix A: Proofs of Propositions 3 and 4

Preliminaries

Let x be a situation such that alternative a is the winner under a run-off point system (f_1 or f_2). Two cases can be distinguished, according to the alternative that is eliminated in the first stage (b or c). In what follows we assume that c is eliminated first (the proofs are similar for the case where b is eliminated first). Given this assumption, it is easily seen that the condition in Proposition 3 reduces to

$$(n_{12} < n/3 \text{ and } n_4 > n_{12} - n_{56} \text{ and } n_{12} + n_2 < n/2) \tag{A1}$$

since, when c is eliminated first under f_1, $n_{56} > n_{12}$ is impossible and $n_{34} > n_{12}$ becomes redundant ($n_{12} < n/3$ implies $n_{34} > n_{12}$). Similarly, the condition in Proposition 4 reduces to

$$(n_{46} > n/3 \text{ and } n_{46} + n_4 > n/2) \tag{A2}$$

since $n_{46} > n_{13}$ is impossible and $n_{46} > n_{25}$ is redundant when c is eliminated first under f_2.

We denote by S_x^b (resp. S_x^c) the set of situations obtained from x by moving down b (c) in at least one individual order. Observe that the definition of S_x^c together with the assumption that c is eliminated first in x imply that, for any $x' \in S_x^c$, c is eliminated first under f_1 as well as under f_2. Consequently, paradox *M2* occurs if and only if there exists some $x' \in S_x^b$ such that b is the winner.

Proof of Proposition 3.

Necessity. Given the above observation, we have to show that, if (A1) does not hold, i.e. if one of the following holds

(α) $n_{12} > n/3$

(β) $n_4 < n_{12} - n_{56}$,

(γ) $n_{12} + n_2 > n/2$,

then $f_1(x') \neq \{b\}$ for any $x' = (n'_1, \ldots, n'_6) \in S_x^b$. Suppose first that ($\alpha$) holds. Since $n'_{12} \geq n_{12}$ for any $x' \in S_x^b$ (with $n'_{ij} = n'_i + n'_j$), it follows from (α) that $n'_{12} > n/3$. Hence, a is not eliminated first in x' under f_1. Moreover, a majority of individuals prefer a to b in x and it is also true for any $x' \in S_x^b$. Hence $f_1(x') \neq \{b\}$ for any $x' \in S_x^b$. Suppose next that (β) holds. Since $n'_{56} \geq n_{56}$ for any $x' \in S_x^b$, we have $n'_{56} \leq n_{56} + n_4$ and the conjunction of this inequality with (β) implies $n'_{56} < n_{12}$; but from the definition of S_x^b, $n'_{12} \geq n_{12}$; hence $n'_{56} < n'_{12}$. Thus, a is not eliminated in the first stage under f_1 and it follows that $f_1(x') \neq \{b\}$ for any $x' \in S_x^b$. Finally, suppose that (γ) holds. Since $n'_{12} \geq n_{12}$ and $n'_2 \geq n_2$, it follows from (γ) that $n'_{12} + n'_2 > n/2$. Now either $n'_{56} \geq n'_{12}$ or $n'_{56} < n'_{12}$. If $n'_{56} \geq n'_{12}$, $n'_{12} + n'_2 > n/2$ implies $n'_{56} + n'_2 > n/2$, i.e. a majority of individuals prefer c to b;

hence, for any $x' \in S_x^b$, b cannot be the winner. If $n'_{56} < n'_{12}$, we obtain the same conclusion.

Sufficiency. Assume that (A1) holds. From x, construct a situation x' in the following way: $n'_j = n_j$ for any $j \in \{1, 2, 3, 5\}$, $n'_4 = n_4 - n_{12} + n_{56} - 1$ and $n'_6 = n_6 + n_{12} - n_{56} + 1$, which is possible since by (A1) $n_4 > n_{12} - n_{56}$. Clearly, $x' \in S_x^b$ (b moves down in at least one individual preference order). We obtain $n'_{12} = n_{12}$; $n'_{34} = n_{34} - n_{12} + n_{56} - 1 = n - 2n_{12} - 1 \geq n_{12}$ since $\sum n_j = n$ and $n_{12} > n/3$ by (A1); and $n'_{56} = n_{56} + n_{12} - n_{56} + 1 = n_{12} + 1$. Consequently, the number of first positions of a is smaller (or equal, but we exclude from consideration tied elections) than the number of first positions of both b and c: a is eliminated. By (A1), we have $n_{12} + n_2 < n/2$, i.e. $n'_{56} + n'_2 \leq n/2$: the number of individuals preferring c to b is smaller than $n/2$, and we conclude that $f_1(x') = \{b\}$. Q.E.D.

Proof of Proposition 4.

Necessity. Suppose that (A2) does not hold: either $n_{46} < n/3$ or $n_{46} + n_4 < n/2$. Noting that $n'_{46} \leq n_{46}$, $n'_4 \leq n_4$ and that a majority of voters prefer a to b for any $x' \in S_x^b$, it is easily checked that if follows from $n_{46} < n/3$ as well as from $n_{46} + n_4 < n/2$ that $f_2(x') \neq \{b\}$ for any $x' \in S_x^b$.

Sufficiency. Suppose that (A2) holds. Observe that this implies that $n_{46} > n_3$: if $n_3 > n_{46}$, then $n_3 + n_4 > n_{46} + n_4$ and we conclude by (A2) that $n_3 + n_4 > n/2$, contradicting the fact that a beats b in the second round. Moreover, $n_{46} > n_3$ implies $n_1 > n_{13} - n_{46}$. Thus from x we can construct a situation x' defined as follows: $n'_1 = n_1 - (n_{13} - n_{46}) - 1$, $n'_2 = n_2 + n_{13} - n_{46} + 1$ and $n'_j = n_j$ for any $j \in \{3, 4, 5, 6\}$. Using the fact that $n_{46} > n/3$ by (A2), it is easily seen that a is eliminated (n'_{46} is higher than both n'_{13} and n'_{25}) and in the second stage b beats c (this follows from $n_{46} + n_4 > n/2$). Q.E.D.

Appendix B

Consider a situation such that a is the winner under f_1. Suppose that an alternative other than a—say b—moves down in some preference orders. Clearly, this change can make b a winner under f_1 only if the numbers of first-place votes are modified, and this implies that b must move down in individual orders where b is ranked first. Hence, any occurrence of *M2* under f_1 implies that the individuals who change their votes prefer b to a. Let us now consider a situation such that a is the winner under f_2. If a moves up in some preference orders, then this move can make a a loser under f_2 only if the number of last-place votes is modified. Consequently, for *M1* to occur under f_2, a must move up in individual orders where a is ranked in last position: the individuals who change their orders prefer any alternative to a.

Appendix C

Observe first that a necessary condition for the occurrence of *M1* and/or *M2* under f_1 and/or f_2 is that there exists an alternative different from the winner which a majority of individuals prefer over the winner. Suppose that $f_1(x) = \{b\}$ or $f_2(x) = \{b\}$; by the above observation and the single-peakedness assumption, *M1* or *M2* occur only if $n_1 > n/2$ or $n_6 > n/2$ (recall that, under the single-peakedness assumption, $n_2 = n_5 = 0$); but this contradicts the fact that b is elected. Hence, neither *M1* nor *M2* can occur when b is the winner and the preferences are single-peaked. Suppose now that $f_1(x) = \{a\}$. As preferences are supposed to be single-peaked, this implies

$$n_1 > n/2 \quad \text{or} \quad (n_{34} < n_1,\ n_{34} < n_6 \text{ and } n_{46} < n/2)\,. \tag{A3}$$

Moreover, we obtain from Proposition 3 that *M2* occurs under f_1 if and only if

$$\begin{aligned}&(n_1 < n/3, n_{34} > n_1 \text{ and } n_4 > n_1 - n_6)\\ \text{or}\quad &(n_6 > n_1, n_6 > n_1 - n_{34} \text{ and } n_1 < n/4)\,. \end{aligned}\tag{A4}$$

It is straightforward to check that (A3) and (A4) cannot both hold. Therefore, *M2* cannot occur under f_1 when a is the winner, and by the symmetry of a and c in the set of situations with single-peaked preferences, the same conclusion holds when c is the winner. Similarly, it can be deduced from Proposition 2 that *M1* never occurs under f_2 when a is the winner and preferences are single-peaked, and the same conclusion holds when c is the winner. We conclude that when single-peakedness is assumed, *M2* never occurs under f_1 and *M1* never occurs under f_2.

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