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Topological Social Choice.

by

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DISCUSSION
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Topological social choice

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Abstract

The topological approach to social choice was developed by Graciela Chichilnisky in the beginning of the eighties. The main result in this area (known as the resolution of the topological social choice paradox) shows that a space of preferences admits of a continuous, anonymous, and unanimous aggregation rule for every number of individuals if and only if this space is contractible. Furthermore, connections between the Pareto principle, dictatorship, and manipulation were established. Recently, Baryshnikov used the topological approach to demonstrate that Arrow's impossibility theorem can be reformulated in terms of the non-contractibility of spheres.
This paper discusses these results in a self-contained way, emphasizes the social choice interpretation of some topological concepts, and surveys the area of topological aggregation.[1]

1 Introduction

Social choice is concerned with providing a rationale for collective decisions when individuals have diverse opinions. Usually this problem is addressed by formalising a set of appealing axioms about how to go from individual to social preferences. The best known model is the Arrowian one: society consists of n individuals, the set A of alternatives is a discrete set ($|A| > 2$), and preferences over A are complete and transitive binary relations.

*This paper is based upon a lecture given at the Pacific Institute for the Mathematical Sciences-workshop on Social Choice and Distributive Justice (PIms, University of British Columbia, Vancouver, July 1998). I want to thank the organizers John Weymark and Hervé Moulin as well as the participants for their stimulating discussion. I am indebted to an anonymous referee, Bart Capéau, and Dirk Scevenels for helpful remarks.

[1] We refer to Chichilnisky (1983) for an earlier survey and to Heal (1997b,c) for a collection of recent papers on topological social choice.

An n-tuple of individual preferences is called a profile. A map from the set of profiles to the set of preferences that satisfies the list of axioms is said to be an aggregation rule. Hence, an aggregation rule transforms profiles into social preferences. Arrow's (1951,1963) theorem shows that his set of axioms (independence, Pareto, universal domain, and non-dictatorship) is incompatible. This negative result revealed the strength of the axioms. Furthermore, only ordinal information is used as an input of the aggregation rule. And finally, the Arrowian model does not provide that many instruments to 'create' a non-dictatorial social preference order out of a profile.

Two decades ago Chichilnisky introduced the topological approach to social choice and caused a major breakthrough in the disentanglement of the possibilities and limitations of preference aggregation in an ordinal framework. Necessary and sufficient conditions to resolve the social choice paradox were established and new insights in the relationships between different aggregation axioms were obtained. The remainder of this introduction gives a preview of the set-up, of the main results, and of the link between the Arrowian and the Chichilnisky model.

In the topological approach the aggregation model becomes more structured: the set A is a subset of $\mathbb{R}^\ell$ and inherits the Euclidean topology, and preferences over A are assumed to be smooth and non-satiated. Such a preference can be represented by means of its normalized gradient field, which assigns to each element a of A a unit vector normal to the tangent space of the indifference surface through a. As such, the domain $\mathcal{P}$ of preferences becomes a subset of the Banach space $V(A), +$ of all smooth vector fields[2] on A and is equipped with (i) a topology, and (ii) the pointwise sum. This additional structure turns out to be extremely useful. First, the topology on $\mathcal{P}$ gives content to the notion of continuity. Second, the addition operator $+$ of vector fields can be used to 'create' new vector fields out of a profile.

Besides enriching the Arrowian model, Chichilnisky imposed a different list of axioms: continuity, anonymity, and unanimity. An aggregation rule that satisfies these axioms is said to be a topological aggregator or a Chichilnisky rule. The domains that allow for a sequence of Chichilnisky rules (one for each number of individuals) are characterized in the 'resolution of the social choice paradox':

Theorem (Chichilnisky and Heal, 1983a)**.**[3] Let $\mathcal{P}$ be a topological space of preferences. Then, there exists for all n a map $F^n : \mathcal{P}^n \to \mathcal{P}$ that is continuous, anonymous, and unanimous if and only if the space $\mathcal{P}$ is contractible.

The condition '$\mathcal{P}$ is contractible' is a topological condition, it is synonymous with 'there are no holes in $\mathcal{P}$'. E.g. convex sets are contractible. Observe that the convex mean is a well defined topological aggregator on a convex subset of $V(A)$.

The space of linear preferences over $A \subset \mathbb{R}^\ell$ has been indicative for the study of topological aggregation. The normalized gradient field of a linear preference is a constant vector

[2] A vector field on A is a map, say p, from A to $\mathbb{R}^\ell$ that assigns to each point $a \in A$ a vector $p(a) \in \mathbb{R}^\ell$.
[3] See section 3.1 for the exact formulation!

field and can be represented by a single point on the sphere $S^{\ell-1} \subset I\!R^{\ell}$ of radius one. The sphere is not contractible and does not admit an aggregation rule. This 'social choice paradox' already appears in Chichilnisky (1979,1980). Nevertheless, the study of continuous maps on spheres has led to further insight in the aggregation axioms. In particular, Pareto-rules, i.e. maps from the profile space $(S^{\ell-1})^n$ to the space $S^{\ell-1}$ of preferences that satisfy continuity and the Pareto principle, were investigated. In this introduction we only mention the following two results.

First, Chichilnisky (1982c) showed that under some additional assumption (called weak positive association condition) the Pareto principle is 'homotopic to' dictatorship. The term 'homotopic to' has a clear meaning in topology: a Pareto-rule can be continuously deformed into a dictatorial rule. In the social choice framework, however, this statement is not that easily comprehensible: it indicates that a single individual plays a dominant role in the determination of the social preference.

In contrast to the previous result, the concept of manipulation is very intuitive: an individual is a manipulator if for any given preferences of his opponents, he can (possibly falsifying his preferences) achieve any desired outcome of the aggregation rule. It turned out that for any Pareto-rule there exists a unique manipulator.

Looking back to one of the roots of social choice, the following question is an intriguing one. Is there any connection between Chichilnisky's result and Arrow's impossibility theorem?

Recently, this question has been answered in the affirmative. In a fascinating article, Baryshnikov (1993) succeeded in unifying both approaches. A reformulation of the combinatorial Arrowian model turns the set of (Arrowian) preferences into a sphere. The Arrowian paradox follows from the non-contractibility of spheres. This link between both approaches is promising but still asks for further investigation.

In conclusion: Chichilnisky's approach to social choice offers an interesting alternative to the traditional combinatorial approach and generated some fundamental insights. Algebraic topology appears to be a rich and powerful tool to study the aggregation of preferences in an ordinal framework.

The larger part of this paper is expository: its purpose is to introduce readers with (almost) no expertise in algebraic topology into this area of social choice theory. In addition, every now and then, some bibliographic notes are inserted. Section 2 summarizes the Arrowian aggregation model, and spells out the technical details of the topological model. Section 3 develops some notions from algebraic topology and repeats the proof of the resolution theorem. Section 4 focusses on linear preferences and proves a fundamental relationship on the degrees of some maps on the sphere. Section 5 focusses on the topological approach to Arrow's theorem. Section 6 has a short look at related literature. Prior to the references section, a short list of textbooks on topology and more specifically on algebraic topology is given.

2 The aggregation problem

2.1 Arrow's theorem

In order to put the Chichilnisky approach into perspective, we summarize Arrow's classical social choice model.

Let A be a discrete set of alternatives and let $N = \{1, 2, \ldots, n\}$ be a set of individuals. Let $\mathcal{Q}$ be the set of all complete and transitive relations on A. A profile $\mathbf{Q} \in \mathcal{Q}^n$ is an n-tuple of individual preferences. Arrow examined aggregation rules of the form

$$H : \mathcal{Q}^n \to \mathcal{Q} : \mathbf{Q} = (Q_1, Q_2, \ldots, Q_n) \mapsto H(\mathbf{Q})$$

and considered the following conditions

- universal domain: the aggregation rule H is defined for all possible profiles of individual preferences,
- weak Pareto principle: if all members strictly prefer a to b, then socially a is strictly preferred to b,
- non-dictatorship: the collective choice is not determined by the choice of a single individual regardless of the preferences of the other individuals,
- independence of irrelevant alternatives: the collective preference on a given pair a, b of alternatives is determined only by the individual preferences on the pair a, b and is not influenced by changing the individual preferences on other alternatives.

The independence axiom is a typical inter-profile condition. Let $\mathbf{P}$ and $\mathbf{Q}$ be two profiles which are related in that their restrictions to a pair a, b of alternatives coincide, then the restrictions of the social preferences $H(\mathbf{P})$ and $H(\mathbf{Q})$ to this pair a, b also coincide. Arrow's impossibility theorem reads:

2.1.1 Theorem (Arrow, 1951,1963)**.** Let there be at least three alternatives. Then, an aggregation rule cannot satisfy the above four conditions.

The independence axiom is looked upon as the culprit of this impossibility result. Indeed, this condition has led to sharper impossibility results. Wilson (1972) dropped the Pareto condition and generalized Arrow's theorem. Campbell and Kelly (1993) state:

> ... even if the Pareto criterion and non-dictatorship are discarded, every social welfare function satisfying the rest of the Arrow's hypothesis is unacceptable.

It appears that every aggregation rule which satisfies Arrow's independence condition either gives some individual too much dictatorial power or else there are too many pairs of alternatives that are socially ranked without consulting anyone's preferences. Furthermore, Malawski and Zhou (1994) have shown that the combination of independence and non-imposition implies the Pareto condition.

2.2 The topological framework

Chichilnisky and Heal (1983a) consider a set $A \subset I\!R^\ell$ of alternatives which is diffeomorphic to the closed unit ball $B^\ell \subset I\!R^\ell$.[4] Preference orders are assumed to be (i) non-satiated and (ii) representable by a utility function (say U) that induces a C^1-gradient field ($p = \nabla U$). This gradient field is then normalized: $\|p(a)\| = 1$ for all $a \in A$. Conversely, if p is a C^1-gradient field and if it satisfies some local integrability conditions, then p is locally the gradient of a utility function of class C^2 (Debreu, 1972). The set $\mathcal{P}$ of all such preferences is a subset of the Banach space $V(A), +$ of C^1-vector fields on A. The space $V(A)$ is equipped with a distance map:

$$d : V(A) \times V(A) \to I\!R^+ : (p, q) \mapsto \sup_{a \in A} \|p(a) - q(a)\|.$$

The topology induced by this distance map is known as the C^1-topology, it makes $\mathcal{P}$ a complete subspace.[5]

The sphere equipped with the Euclidean topology is a particular case of this more general framework. The sphere coincides with the set of linear preferences. Indeed, the normalized gradient fields are then constant and can be represented by a point on the unit sphere. As spheres are the prototypes of non-contractible sets, they play an important role in the Chichilnisky framework. Section 4 returns to this issue.

Finally, it is important to observe that due to the normalisation of the gradient fields no distinction is made between the profiles

$$(U_1, U_2, \ldots, U_n) \quad \text{and} \quad (f_1 \circ U_1, f_2 \circ U_2, \ldots, f_n \circ U_n)$$

with $f_1, f_2, \ldots, f_n$ strictly increasing C^2-functions. Hence, the aggregation is done in an ordinal non-comparable framework.

2.3 Chichilnisky rules

Let $\mathcal{P}$ be a set of preferences equipped with a topology. A topological aggregation rule or Chichilnisky rule is a map

$$F : \mathcal{P}^n \to \mathcal{P} : \mathbf{P} = (P_1, P_2, \ldots, P_n) \mapsto F(\mathbf{P})$$

that satisfies

[4]Hence, there exists a differentiable one-to-one correspondence

$$\varphi : A \to B^\ell = \{x \in I\!R^\ell |\ \|x\|^2 = x_1^2 + \ldots + x_\ell^2 \leq 1\}$$

for which the inverse map $\varphi^{-1} : B^\ell \to A$ is also differentiable.

[5]For a discussion on this particular topology we refer to Allen (1996), Baigent (1997), Baigent and Huang (1990), Chichilnisky (1982a,b,1991,1996a,1997), Chichilnisky and Heal (1983a), Heal (1997a), Le Breton and Uriarte (1990a,b), and Uriarte (1987). Chichilnisky and Heal observe that the only condition on the topology on $\mathcal{P}$ for the validity of their results, is that the space of linear preferences inherits the Euclidean topology.

- anonymity: for all permutations $\sigma : N \to N$ and for all profiles $\mathbf{P} \in \mathcal{P}^n$ we have

$$F(P_1, P_2, \ldots, P_n) = F(P_{\sigma(1)}, P_{\sigma(2)}, \ldots, P_{\sigma(n)}),$$

- unanimity: for any preference $P \in \mathcal{P}$ we have $F(P, P, \ldots, P) = P$,
- continuity: the map F is continuous; the domain $\mathcal{P}^n$ is hereby equipped with the product topology.

Anonymity demands that the collective choice remains unaltered whenever individuals exchange their preferences. Obviously, anonymity is stronger than Arrow's non-dictatorship axiom.

Unanimity requires that for a profile of identical individual preferences over A the social preference order coincides with the individual preference order. Unanimity is weaker than the Pareto condition. Note that unanimity implies a non-imposition property: $\mathrm{Im}(F) = \mathcal{P}$.

Continuity replaces Arrow's independence axiom as the inter-profile consistency condition. If two profiles of individual preferences are close, then the resulting social preferences are required to be close. Where Arrow's independence expresses stability of the collective preference on each pair of alternatives under changes in the rankings of other alternatives, continuity expresses stability of the collective preference order under small changes in the profile. Concerning the relationship between both inter-profile axioms, Chichilnisky (1982a) indicates that neither of them implies the other.

2.3.1 Remark. Chichilnisky's continuity demand is controversial. Continuity can be imposed with impunity if the topology is sufficiently large: with respect to the discrete topology every map is continuous. Baigent and Huang (1990) and Baigent (1997) appeal to underpin the continuity demand together with the particular choice of the topology by means of fundamental principles.[6]

Arrow considered a discrete choice set (e.g. a set of candidates). A kind of dichotomy is at issue: candidates are either different and easy to distinguish or equal. If people rank some candidates differently, they are said to have different preferences. The same dichotomy shows up: preferences are either different and easy to distinguish or equal. Continuity of the preferences and continuity of the aggregation rule is an empty device then. To state it differently: in this case the discrete topology seems 'natural'.

Things change dramatically, when the choice set is some connected subset of Euclidean space (the natural shelter for economic models). Empirically, it becomes impossible to distinguish different alternatives if the Euclidean distance between them is sufficiently small. In this case it is natural that if alternative a is judged to be superior to b then anything sufficiently close to a should be superior to anything sufficiently close to b. Continuity of

[6]Chichilnisky's (1991, section 3) attitude in this discussion is more pragmatic: social choice will benefit from an integration into the general body of economic theory and if topological techniques happen to fit into this program then one should not hesitate to exploit them.

the social and of the individual preferences becomes inevitable.[7] These continuity conditions are not innocent: Arrow's independence axiom in combination with continuity of the social and of the individual preferences, guides us to unattractive aggregation rules such as dictatorship or inverse dictatorship (Campbell, 1992b).

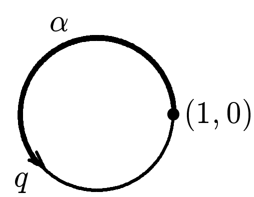
Now, take the next step. Assume that preferences are smooth or even (locally) linear. Then the set of preferences becomes a subset of a topological (normed) space, and preferences itself can become empirically indistinguishable. In this setting, it is natural to require that the collective choice process is not unduly sensitive to errors in reporting or observing the individual preferences. Chichilnisky's continuity axiom tries to capture this idea. Of course, this presupposes some care in the choice of the topology as demonstrated by Le Breton and Uriarte (1990) (cf. remark 4.2.3). Note however that in case the space $\mathcal{P}$ of preferences is a subset of an Euclidean space, it inherits the Euclidean topology.

Finally, it is the Euclidean topology that makes Baryshnikov's unification approach possible. In his technical framework the independence condition forces the corresponding aggregation map to be continuous with respect to the Euclidean topology.

2.3.2 Examples. Before introducing the resolution theorem, we provide some examples of aggregation maps on the unit circle S^1 equipped with the Euclidean topology that satisfy all but one of Chichilnisky's axioms. Of course, as the circle is non-contractible these maps cannot satisfy all of the axioms.

- Let $q \in S^1$. The constant map $S^1 \times \ldots \times S^1 \to S^1 : \mathbf{p} \mapsto q$ is continuous and anonymous.

- Represent the circle S^1 by the interval $[0, 2\pi[$, i.e. a point q on the circle is represented by the positive angle α measured from the point $(1,0) \in S^1$. Then the following maps



$$\begin{cases} (\alpha_1, \ldots, \alpha_n) \mapsto \frac{1}{n}(\alpha_1 + \ldots + \alpha_n), \\ (\alpha_1, \ldots, \alpha_n) \mapsto \text{median}\,\{\alpha_1, \ldots, \alpha_n\} \end{cases}$$

  are anonymous and unanimous. Continuity is violated: points close to but at different sides of $(1,0)$ generate divergent outcomes. Similarly, the Pareto condition is violated.

  In case the circle is restricted to its positive part (i.e. $\alpha \in [0, \pi/2]$), the second map satisfies anonymity, the Pareto condition, continuity, and Arrow's independence axiom (Nitzan, 1976).

- The map

$$S^1 \times S^1 \to S^1 : (p_1, p_2) \mapsto \nu\left(\|p_1 - p_2\|\, p_1 \;+\; (2 - \|p_1 - p_2\|)\, p_2\right),$$

  where ν normalizes a non-zero vector, i.e. $\nu(q) = q/\|q\|$, satisfies continuity, weak Pareto (hence unanimity), and non-dictatorship, and violates anonymity.

[7] Campbell (1992a, chapter 2) provides convincing examples.

3 The resolution theorem

3.1 Statement and definitions

We repeat the resolution theorem and explain the topological concepts that appear in its statement.

3.1.1 Theorem (Chichilnisky and Heal, 1983a). Let the preference space $\mathcal{P}$ be a path-connected parafinite CW-complex. Then a necessary and sufficient condition for the existence of a Chichilnisky rule for each number of individuals is that $\mathcal{P}$ is contractible.

First, we explain the notion of CW-complex.[8] Such a complex refers to a topological space that is built in stages, each stage being obtained from the preceding by adjoining cells of a given dimension. Hereby, a topological space Y is said to be obtained from X by **adjoining** ℓ-cells (with $\ell \in \mathbb{N}_0$) if $Y - X$ is the disjoint union of open subsets e^ℓ_λ (with λ running over an index set Λ_ℓ) each of which is homeomorphic to the ℓ-dimensional open ball $\dot{B}^\ell \subset \mathbb{R}^\ell$, i.e. there exists a continuous one-to-one correspondence

$$\varphi^\ell_\lambda : \dot{B}^\ell = \{x \in \mathbb{R}^\ell |\ \|x\|^2 = x_1^2 + \ldots + x_\ell^2 < 1\} \ \longrightarrow\ e^\ell_\lambda$$

for which the inverse map $(\varphi^\ell_\lambda)^{-1} : e^\ell_\lambda \to \dot{B}^\ell$ is also continuous. There are no restrictions on the cardinalities of the index sets $\Lambda_1, \Lambda_2, \ldots$; they might be empty. It is also assumed that for each ℓ and each $\lambda \in \Lambda_\ell$ the map φ^ℓ_λ extends to a continuous map

$$f^\ell_\lambda : B^\ell = \{x \in \mathbb{R}^\ell |\ \|x\|^2 = x_1^2 + \ldots + x_\ell^2 \leq 1\} \ \longrightarrow\ \overline{e}^\ell_\lambda$$

with $\overline{e}^\ell_\lambda$ the closure of e^ℓ_λ, that maps $S^{\ell-1} = B^\ell - \dot{B}^\ell$ into X.

3.1.2 Definition. A Hausdorff space X is said to be a CW**-complex** if it can be expressed $X = \cup_{k=0}^\infty X_k$ satisfying

- $X_0 \subset X_1 \subset \ldots \subset X_k \subset \ldots,$
- X_0 is a discrete topological space,
- X_{k+1} with $k \in \mathbb{N}$ is obtained from X_k by attaching a collection of $k+1$-cells,
- X is supposed to have the weak topology: a subset A of X is closed if and only if $A \cap \overline{e}^\ell_\lambda$ is closed (and compact) for each cell e^ℓ_λ.

A complex X is said to be finite if only a finite number of cells are involved. It is said to be of dimension k if in the above construction $X = X_k$. A finite CW-complex of dimension k can be embedded in $\mathbb{R}^{2k+1}$. Finally, X is said to be parafinite if, for every k, there are only a finite number of k-cells. A parafinite complex can be embedded in the countably infinite dimensional Euclidean space $\mathbb{R}^\infty$.

3.1.3 Properties. Maunder (1996, Ch 7) observes that

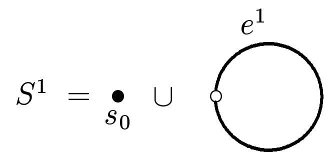
[8] With respect to this limitation, Heal (1983) and Baryshnikov (1997) argue that the class of CW-complexes is large enough to cover most if not all spaces of preferences that arise naturally in economics. On the other hand, Horvath (1995) points at some shortcomings of this class and lists sufficient conditions for a larger class of topological spaces in order to admit Chichilnisky rules.

- a CW-complex is closure finite: for each cell e^ℓ_λ the intersection of all subcomplexes containing this cell is a finite subcomplex,[9]

- for two cells e^ℓ_λ and e^m_μ we have $\varphi^\ell_\lambda(\dot{B}^\ell) \cap \varphi^m_\mu(\dot{B}^m)$ is empty unless $\ell = m$ and $\lambda = \mu$,

- in general the product of two CW-complexes is not a CW-complex.

3.1.4 Examples. (i) Let us verify that the ℓ-sphere $X = S^\ell$ is a CW-complex. Let $X_0 = \{s_0 = (-1, 0, \ldots, 0)\}$ be a singleton and add one ℓ-cell. It is sufficient to consider a continuous map $f^\ell : B^\ell \to X$ that maps the boundary $S^{\ell-1}$ into the point s_0. (ii) All smooth ℓ-manifolds, all polyhedra, and spaces homeomorphic to a polyhedron are CW-complexes. (iii) The comb space CS, defined by



$$CS = \left\{(x, y) \in I\!R^2 \,|\, 0 \le x, y \le 1 \quad \text{and} \quad y \neq 0 \text{ implies } x \in \{0, 1, 1/2, 1/3, \ldots\}\right\},$$

and embedded in the Euclidean plane is not a CW-complex. Indeed, the set X_0 of zero-cells contains the collection $\{(0,1), (1,1), (1/2,1), \ldots, (1/n,1), \ldots\}$ which is not discrete.

Next, we clarify the contractibility condition.

3.1.5 Definition. A topological space X is said to be **contractible** if there exists a point $x_0 \in X$ and a continuous map $\vartheta : X \times I \to X$ with $I = [0,1]$ such that

$$\forall x \in X : \vartheta(x, 0) = x \quad \text{and} \quad \vartheta(x, 1) = x_0.$$

In words, contractibility means that the identity map can be continuously deformed to a constant map. This concept of continuous deformation can be generalized to arbitrary maps.

Let X, Y be two topological spaces. Two continuous **maps** $f, g : X \to Y$ are said to be **homotopic** $(f \simeq g)$ if there exists a continuous map

$$\vartheta : X \times I \to Y$$

such that $\vartheta(\,.\,, 0) = f(\,.\,)$ and $\vartheta(\,.\,, 1) = g(\,.\,)$. Hence, X is contractible if there exists a point $x_0 \in X$ such that the maps

$$Id : X \to X : x \mapsto x \quad \text{and} \quad C_{x_0} : X \to X : x \mapsto x_0$$

are homotopic.

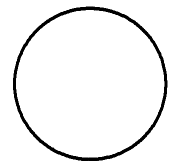
Two **spaces** X, Y are said to be **homotopic** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. In this case the map f (and g) is said to be a **homotopy equivalence**. The relation 'is homotopic to' induces an equivalence

[9] The initials CW stand for Closure finite and Weak topology (e.g. Gray, 1975).

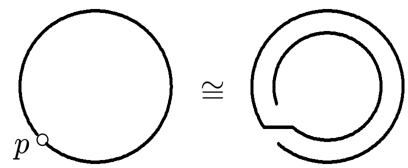
relation on the class of all topological spaces, the equivalence classes are called homotopy types. Observe that homeomorphic spaces belong to the same homotopy type, the converse is not true.[10] Also, a space is contractible if and only if it has the same homotopy type as a one-point space (e.g. Spanier, 1966).

3.1.6 Examples. (*i*) A convex set $X \subset I\!R^\ell$ is contractible: define $\vartheta(x,t) = (1-t)x + tx_0$ with $x_0 \in X$ arbitrary and $t \in I$.

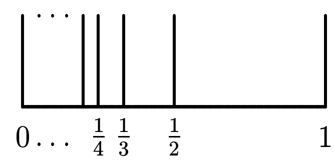
(*ii*) The circle S^1 is not contractible. More general, the sphere $S^{\ell-1}$ is not contractible (cf. section 3.2).



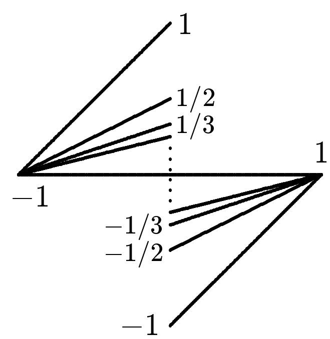
(*iii*) When one single point p is dropped, the remaining space $S^1 - p$ is contractible. The second space is homeomorphic to the first one and is therefore contractible.



(*iv*) The comb space is not a CW-complex but is contractible. The map $\vartheta : CS \times I \to CS : ((x,y),t) \mapsto (x,(1-t)y)$ is a homotopy from the identity to the projection to the x-axis. This projection is homotopic to a constant map.

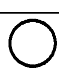
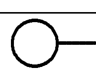


(*v*) The double comb space is not a CW-complex and is not contractible (Maunder, 1996, Ex 7 5 5). To provide some intuition: the central point $(0,0)$ is the limit of the sequences $\{(0,\frac{1}{k})\}$ and $\{(0,-\frac{1}{k})\}$ and a continuous deformation of this space into one single point pushes this central point to the left extreme $(-1,0)$ as well as to the right extreme $(1,0)$.



3.2 How to prove the resolution theorem?

This subsection develops a strategy to prove the 'only if part' of theorem 3.1.1: if $\mathcal{P}$ admits topological aggregators then $\mathcal{P}$ is contractible. The proof is not constructive (in the sense that a contraction map ϑ is defined) but follows an algebraic topological route. The intuition is as follows: contractibility of a space means that there are no holes in it, an n-dimensional hole can be detected through the non-triviality of the nth homotopy group, hence in order to prove contractibility one has to show that all homotopy groups are trivial.

[10]The spaces  and  are homotopic but not homeomorphic.

We start with the definition of closed paths and of the first homotopy group.
Let X be a topological space and let $x_0 \in X$. A **closed path at** x_0 in X is a continuous map $\alpha : I \to X$ for which $\alpha(0) = \alpha(1) = x_0$. Two paths are considered equivalent if they are homotopic. The equivalence class containing the closed path α is denoted by $[\alpha]$. Next, we define an operator upon the set of paths: given a couple of paths, how to create a new one? The construction goes as follows. Let α and β be two closed paths, the new closed path $\alpha \cdot \beta$ travels with a doubled speed first along α, then along β:

$$\alpha \cdot \beta : I \to X : t \mapsto \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2, \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

This product respects the above defined equivalence: when $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$.

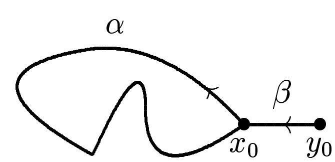
3.2.1 Definition. The **first homotopy group** or fundamental group $\pi_1(X, x_0), \cdot$ of the space X is the set of homotopy classes of closed paths in X at x_0 equipped with the operation '$\cdot$'. The neutral element is the class $e = [\gamma_0]$ that contains the constant path γ_0, i.e. $\gamma_0(t) = x_0$ for all $t \in I$; and the inverse of a class $[\alpha]$ is the class of the closed path α^{-1} defined by

$$\alpha^{-1} : I \to X : t \mapsto \alpha(1-t),$$

i.e. travelling backwards along α.

3.2.2 Properties.

- The group $\pi_1(X, x_0)$ depends upon the point x_0, but in case the space X is path-connected the reference to the base point x_0 becomes redundant and all the first homotopy groups are isomorphic. Indeed, a closed path $\alpha \in \pi_1(X, x_0)$ corresponds to $\beta \cdot \alpha \cdot \beta^{-1} \in \pi_1(X, y_0)$.



- Let X, Y be two topological spaces. Then, a continuous map $f : X \to Y$ induces a homomorphism[11] $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) : [\alpha] \mapsto [f \circ \alpha]$.

- Also, the homotopy group $\pi_1(X \times Y)$ of the product is isomorphic to the product $\pi_1(X) \times \pi_1(Y)$ of the homotopy groups.

- Finally, homotopic spaces have isomorphic homotopy groups.

3.2.3 Examples. (i) A contractible space is homotopic to a point and has a trivial fundamental group. (ii) The fundamental group $\pi_1(S^1)$ of a circle is isomorphic to $\mathbb{Z}$. Indeed, consider the closed path

$$\alpha : t \mapsto (\cos 2\pi t\, ,\, \sin 2\pi t).$$

[11] Let $G_1, \cdot$ and $G_2, \cdot$ be two groups. A homomorphism is a map $h : G_1 \to G_2$ that satisfies $h(a \cdot b) = h(a) \cdot h(b)$ for all $a, b \in G_1$. If, in addition, h is a one-to-one correspondence, h is said to be an isomorphism.

Any closed path in S^1 is homotopic to a multiple of α or α^{-1}. The element $[\alpha]$ is said to be generator of $\pi_1(S^1)$ and $\pi_1(S^1) \cong \mathbb{Z}$.[12] For the higher dimensional spheres $S^2, S^3, \ldots$ the fundamental groups are trivial.
Intuitively: a space has a non-trivial fundamental group as soon the space exhibits one-dimensional holes, the number of which is reflected in the number of generators of the fundamental group.

In order to study higher dimensional holes, higher homotopy groups have been designed. Let $\ell \in I\!N_0$, let X be a topological space, and let $x_0 \in X$. Consider the set $\pi_\ell(X, x_0)$ of homotopy classes of continuous maps from $I^\ell \to X$ that map the border $\partial I^\ell = I^\ell - \dot{I}^\ell$ into x_0. Again, two classes $[\alpha]$ and $[\beta]$ in $\pi_\ell(X, x_0)$ can be multiplied: the class $[\alpha] \cdot [\beta]$ contains the map

$$\alpha \cdot \beta : I^\ell \to X : (t_1, t_2, \ldots, t_\ell) \mapsto \begin{cases} \alpha(2t_1, t_2, \ldots, t_\ell) & 0 \leq t_1 \leq 1/2, \\ \beta(2t_1 - 1, t_2, \ldots, t_\ell) & 1/2 \leq t_1 \leq 1. \end{cases}$$

This multiplication is well defined and turns $\pi_\ell(X, x_0), \cdot$ into a group: the ℓth homotopy group. In addition to the properties in the previous list (3.2.2), the higher homotopy groups are abelian.

3.2.4 Examples. (i) All homotopy groups of a contractible space are trivial. (ii) For a sphere, we have $\pi_\ell(S^\ell) \cong \mathbb{Z}$: the 'one' ℓ-dimensional hole in S^ℓ corresponds to a non-trivial ℓth homotopy group with 'one' generator. Furthermore, $\pi_k(S^\ell) = \{0\}$ for $k < \ell$.[13]

Observe that homotopic spaces generate isomorphic homotopy groups. The converse statement, alas, is not true: all homotopy groups of the double comb space are trivial (Maunder, 1996, Ex 7 5 5) and yet this space is not homotopic to a point. In case of CW-complexes however, life becomes much easier:

3.2.5 Theorem (JHC Whitehead, 1949)**.** Let $f : X \to Y$ be a continuous map between two path-connected CW-complexes X and Y. If f induces isomorphisms $f_* : \pi_\ell(X) \to \pi_\ell(Y)$ for all ℓ, then f is a homotopy equivalence.
Proof. See Maunder (1996, Thm 7 5 4).

As a matter of fact this theorem makes the family of CW-complexes the most tractable family for homotopy theory. In addition, for every path-connected topological space X there exists a CW-complex K and a continuous map $f : K \to X$ that induces isomorphisms at each homotopy level. The following lemma, a consequence of the previous theorem, summarizes our strategy in proving the 'only if' part of the resolution theorem.

3.2.6 Lemma (Maunder, 1996, Corr 8 3 11)**.** Let X be a path-connected CW-complex. Then X is contractible if and only if all homotopy groups $\pi_1(X), \pi_2(X), \ldots, \pi_k(X), \ldots$ are trivial.

[12] From the fact that the fundamental groups of S^1 and B^2 are different, it follows that S^1 and B^2 are not homotopic. This implies the two-dimensional version of Brouwer's fixed point theorem (e.g. Massey, 1977).
[13] The statement "$\pi_k(S^\ell) = \{0\}$ for $k > \ell$" is false: e.g. $\pi_5(S^2) \cong \mathbb{Z}_2$ (Spanier, 1966).

3.2.7 Remark. In view of certain pathological spaces such as the double comb space, it is only in terms of CW-complexes that the statements 'being contractible' and 'having no holes' are synonymous.

3.3 Proof of the resolution theorem

The 'only if part'.
Let $\mathcal{P}$ be a path-connected parafinite CW-complex, and let $F^2, F^3, \ldots, F^n, \ldots$ be a sequence of Chichilnisky rules (one for every $n \geq 2$). Hence, for all $k \in I\!N_0$ there exists a sequence of homomorphisms

$$F_*^n : \pi_k(\mathcal{P}^n) = \underbrace{\pi_k(\mathcal{P}) \times \pi_k(\mathcal{P}) \times \ldots \times \pi_k(\mathcal{P})}_{n \text{ times}} \to \pi_k(\mathcal{P}),$$

with $n = 2, 3, \ldots$. Note that we used property 3.2.2: the homotopy group of a product is the product of the homotopy groups.
In view of lemma 3.2.6, it is sufficient to show that all the homotopy groups are trivial, i.e. $\pi_k(\mathcal{P}) = \{0\}$ for all $k \in I\!N_0$. This is proven by contradiction.
Suppose that $\pi_1(\mathcal{P}) = \ldots = \pi_{\ell-1}(\mathcal{P}) = \{0\}$ and that $\pi_\ell(\mathcal{P})$ is the lowest non-trivial homotopy group. It will turn out that the group $\pi_\ell(\mathcal{P})$ is abelian (Step 1) and finitely generated (Step 2). Group theory provides us with a full classification of these objects, and the desired contradiction will be obtained (Step 3):

Step 1. The homotopy groups of level two or higher are abelian by definition. That also the first homotopy group $\pi_1(\mathcal{P})$ is abelian follows from the existence of one single Chichilnisky rule (say, F^n). Indeed, let a and b be two classes in $\pi_1(\mathcal{P})$, e is the neutral element. By unanimity and anonymity we have[14]

$$\begin{aligned} a = F_*^n(a, a, \ldots, a) &= F_*^n(a, e, \ldots, e) \cdot F_*^n(e, a, e, \ldots, e) \cdot \ldots \cdot F_*^n(e, e, \ldots, e, a) \\ &= n \times F_*^n(a, e, \ldots, e) \\ &= F_*^n(n \times a, e, \ldots, e). \end{aligned}$$

Hence,

$$\begin{aligned} a \cdot b &= F_*^n(n \times a, e, \ldots, e) \cdot F_*^n(n \times b, e, \ldots, e) \\ &= F_*^n(n \times a, e, \ldots, e) \cdot F_*^n(e, n \times b, e, \ldots, e) \\ &= F_*^n(n \times a, n \times b, e, \ldots, e) \\ &= F_*^n(n \times b, n \times a, e, \ldots, e) \\ &= b \cdot a. \end{aligned}$$

As a consequence, we are allowed to write the homotopy groups as additive groups: $\pi_k(\mathcal{P}), +$ with 0 the neutral element, and $-a$ the inverse of a. Observe that the above calculations can be done at any homotopy level: the identity

$$g = n \times F_*^n(g, 0, \ldots, 0) = \overbrace{F_*^n(g, 0, \ldots, 0) + \ldots + F_*^n(g, 0, \ldots, 0)}^{n \text{ times}} \tag{1}$$

[14] We write $k \times a$ for $\overbrace{a \cdot a \cdot \ \ldots \ \cdot a}^{k \text{ times}}$.

holds for all $g \in \pi_k(\mathcal{P})$ and for all $k \in I\!N_0$. The existence of an n-rule on $\mathcal{P}$ implies that multiplication by n is a group isomorphism on all homotopy levels.

Step 2. Since $\mathcal{P}$ is parafinite, Hurewicz theorem implies that $\pi_\ell(\mathcal{P})$ is finitely generated,[15] i.e. there exists a finite subset $\{g_1, g_2, \ldots, g_m\} \subset \pi_\ell(\mathcal{P})$ of generators such that any g in $\pi_\ell(\mathcal{P})$ can be expressed (not necessary in a unique way) in the form

$$g = z_1 g_1 + z_2 g_2 + \ldots + z_m g_m,$$

with $z_1, z_2, \ldots, z_m \in \mathbb{Z}$. The order of such a generator g_k is either infinite, i.e. for different integers z and z' in $\mathbb{Z}$ the elements zg_k and $z'g_k$ in $\pi_\ell(\mathcal{P})$ are different; either finite, i.e. there exists an $m \in I\!N_0$ for which $mg_k = 0$.

Step 3. Let the space $\mathcal{P}$ admit an n-rule for any natural number. Let g be a generator of the lowest non-trivial homotopy group $\pi_\ell(\mathcal{P})$.

Suppose that g is of infinite order. The group $G\{g\}$ generated by g is isomorphic to $\mathbb{Z}$. Equation (1) states that g is divisible by n for all $n = 2, 3, \ldots$. But then g has to be 0 because 0 is the only element in $\mathbb{Z}$ that can be divided by any integer. Hence, the group $\pi_\ell(\mathcal{P})$, being abelian, only contains elements of finite order and is therefore a finite group. Next, suppose that $g \neq 0$ is of finite order and let $m \geq 2$ be the smallest natural number such that $mg = 0$. Let a satisfy equation (1) with $n = m$, i.e. $g = ma$. In case the group $G\{a\}$ generated by a and the group $G\{g\}$ generated by g coincide, it follows that a is a multiple of g. But then we have $ma = 0$, this is in contradiction with $g \neq 0$. In case the groups $G\{a\}$ and $G\{g\}$ do not coincide, then $G\{g\} \subset G\{a\}$. Hence, in the set of generators the element g can be dropped and a might be added. Then, apply the previous reasoning upon the generator a. As $\pi_\ell(\mathcal{P})$ is a finite group, this process ends in the desired contradiction. Conclude that $\pi_\ell(\mathcal{P}) = \{0\}$. Hence, all homotopy groups are trivial and $\mathcal{P}$ is contractible.

The 'if part' (there exist Chichilnisky rules *'if'* the space of preferences is contractible). Conversely, let $\mathcal{P}$ be a contractible parafinite CW-complex. If $\mathcal{P}$ is convex (in $I\!R^\infty$), then the convex mean

$$F^c : (P_1, \ldots, P_n) \mapsto \tfrac{1}{n}(P_1 + \ldots + P_n)$$

defines an n-rule for all $n \geq 2$. Otherwise, let $K(\mathcal{P})$ be the convex hull of $\mathcal{P}$ in $I\!R^\infty$. Then $K(\mathcal{P})$ is constructed by adding a finite number of new cells of each dimension and is a CW-complex with $\mathcal{P}$ as a subcomplex. Since both spaces $\mathcal{P}$ and $K(\mathcal{P})$ are contractible, the inclusion map $i : \mathcal{P} \to K(\mathcal{P})$ is a homotopy equivalence (use theorem 3.2.5). This implies the existence of a continuous map $r : K(\mathcal{P}) \to \mathcal{P}$ for which the restriction $r|_\mathcal{P}$ is equal to the identity map on $\mathcal{P}$ (e.g. Lundell and Weingram, 1969, Thm IV 3 1; or Rohlin and Fuchs, 1981, p119).[16] Now, the following maps are well defined and satisfy all the

[15] Hurewicz theorem states that in case all homotopy levels up to $\ell - 1$ are trivial, then the ℓth homotopy group is isomorphic to the ℓth homology group (Maunder, 1996, Th 8 3 7). Since the homology groups of a path-connected parafinite CW-complex are finitely generated, the assertion follows.

[16] To provide some intuition: as $\mathcal{P}$ is contractible the convex hull is constructed from $\mathcal{P}$ through attaching cells 'on the outside' of $\mathcal{P}$. The map r has to push these new cells towards the border of $\mathcal{P}$. See remark 3.4.6 for an illustration.

Chichilnisky axioms:

$$\mathcal{P}^n \xrightarrow{i} K(\mathcal{P})^n \xrightarrow{F^c} K(\mathcal{P}) \xrightarrow{r} \mathcal{P} : \mathbf{P} \mapsto r \circ F^c(\mathbf{P}),$$

for $n \geq 2$.

3.4 Remarks

3.4.1 Some of the ideas in the above proof also appear in Eckman (1954) and in Candeal and Induráin (1995). Eckman studied topological n-means and proved the resolution theorem for polyhedra (a special class of CW-complexes). In his article, however, there is no sign of a social choice interpretation. The proof in Candeal and Induráin avoids Hurewicz theorem (footnote 15), but needs a more general classification result of abelian groups.

3.4.2 Baigent (1984,1985) suggests a slightly different approach. He stresses the anonymity condition and considers profiles in $\mathcal{P}^n$ that are identical up to a permutation as equivalent. Let $[\mathbf{P}]$ denote the (unique) equivalence class that contains the profile $\mathbf{P}$ and its permutations. Obviously, it is sufficient to define a rule $\overline{F}$ on the quotient space $\overline{\mathcal{P}^n}$ of these equivalence classes. In order to give content to the notion of continuity, this quotient space needs to be equipped with a topology. A natural candidate is the quotient topology, i.e. the largest topology on $\overline{\mathcal{P}^n}$ that makes the map $\pi : \mathcal{P}^n \to \overline{\mathcal{P}^n} : \mathbf{P} \mapsto [\mathbf{P}]$ continuous. The whole setup generates a commutative diagram:

$$\begin{array}{ccc} \mathcal{P}^n & \xrightarrow{F} & \mathcal{P} \\ \downarrow \pi & & \downarrow Id \\ \overline{\mathcal{P}^n} & \xrightarrow{\overline{F}} & \mathcal{P}, \end{array}$$

i.e. each continuous and anonymous map F defines a continuous map $\overline{F}$ and vice versa. As a consequence, the original Chichilnisky approach and this alternative approach both produce the same results (Le Breton and Uriarte, 1990b and Zhou, 1997).

3.4.3 The proof of the resolution theorem does not use any additional properties of the space $\mathcal{P}$ and its topology except that it is a path-connected parafinite CW-complex (Chichilnisky, 1996a). The model presented in section 2.1 only serves as an example of how to construct such a space.

3.4.4 The contractibility condition in the resolution theorem provides a domain restriction for which the topological aggregation problem is solvable. It tests whether there is a way of 'deforming continuously' the space of preferences into a single point (or preference). Such a continuous deformation, however, might be rather wild and interpreting such transformations might be difficult.

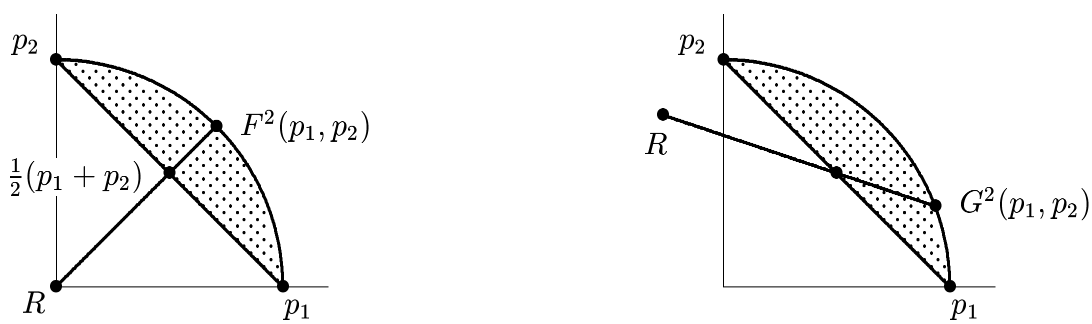
Heal (1983) interprets contractibility in terms of some sort of limited agreement. E.g. in the contractible space obtained from the sphere after deleting one single point (cf. 3.1.6 *iii*: $S^1 - p$) there is a limited agreement: individuals agree on not moving in that particular direction. It seems hard to extend this interpretation to an arbitrary contractible space (cf. 3.1.6 *iii*: the spiral).

3.4.5 Observe that step 3 in the proof of the 'only if part' is valid as soon there exists a topological aggregation rule for any prime number.

Chichilnisky (1996b) isolates the implications of the existence of one single n-rule with n a prime number: it follows that all the homotopy groups consist out of elements of finite order relatively prime to n. See also Candeal et al (1992) and Candeal and Induráin (1995).

3.4.6 In order to illustrate the 'only if part' in the proof, we construct a topological aggregation rule on the space ML of monotone linear preferences. Linearity implies constant gradient fields and monotonicity implies that the gradient vector is positive. Consequently, ML can be represented as the positive part of a sphere and is contractible.

The following picture shows the situation in the two-dimensional two-person case. The convex hull $K(ML)$ is obtained from ML after attaching one 1-cell (the line segment $]p_1, p_2[$) and one 2-cell (the interior shaded area). There are many maps from the convex hull $K(ML)$ to ML that extend the identity map on ML. There are as many Chichilnisky rules on ML. The picture considers projections $r : K(ML) \to ML$ with base point R. The aggregated preferences of the profile (p_1, p_2) are drawn. Observe that the second rule G^2 is 'biased towards' the preference p_1.



Construction of a rule starting from the convex mean on $K(ML)$.

3.4.7 Some spaces $\mathcal{P}$ of preferences allow for an alternative construction of a topological aggregator. The idea is to make use of a continuous identification map $\beta : \mathcal{P} \to U$ where U is a certain class of utility functions. In case the selection β has a continuous projection map $\pi : U \to \mathcal{P} : u \mapsto P$ defined by $(a, b) \in P$ if and only if $u(a) \geq u(b)$, the following diagram appears:

$$\begin{array}{ccc} \mathcal{P}^n & \xrightarrow{F} & \mathcal{P} \\ \downarrow \beta & & \uparrow \pi \\ U^n & \xrightarrow{F^U} & U, \end{array}$$

where β is extended to $\mathcal{P}^n$: $\beta(P_1, P_2, \ldots, P_n) = (\beta(P_1), \beta(P_2), \ldots, \beta(P_n))$. Obviously, an aggregation rule F^U defined on the space U induces an aggregation rule F on the space $\mathcal{P}$. As the construction of F is based upon the selection map β only ordinal information is used.

The existence of such a diagram and an aggregation rule F^U was proved for the following spaces of preferences. The set $A \subset \mathbb{R}^\ell$ of alternatives is assumed to be convex and compact.

- The space of continuous and strictly Schur-convex preferences, i.e. continuous preferences P such that for all $\mathbf{x} \in \mathbb{R}^k$ with $\sum x_i$ fixed and for all bistochastic matrices B we have: $(\mathbf{x}, B\mathbf{x}) \in P$ implies that B is a permutation matrix, equipped with a suitable topology (Le Breton et al, 1985).
- The space $\mathcal{P}_{\text{sco}}$ of strictly quasi-concave preferences (a subspace of the space of all continuous preferences on A endowed with the topology closed convergence (Le Breton and Uriarte, 1990a; Chichilnisky, 1991).
- The space of continuous monotone preferences (Allen, 1996; see also Chichilnisky and Heal, 1983a, Ex 1; Chichilnisky, 1996a).

Note that $\mathcal{P}_{\text{sco}}$ allows for thick indifference curves and satiation points. Remark 4.2.3 returns to this issue.

3.4.8 Candeal et al (1992), Efimov and Koshevoy (1994), and Horvath (1995) study Chichilnisky rules on other structures such as topological vector-spaces, semi-lattices, metric spaces, ... We list three simple examples of spaces admitting a Chichilnisky rule although they are not contractible or not a CW-complex:

- $\mathcal{P}$ = the set of rational numbers and F = convex mean,
- $\mathcal{P}$ = the set of irrationals and F = minimum,
- $\mathcal{P}$ = the comb space CS and $F\left((x_1, y_1), \ldots, (x_n, y_n)\right) = (\min_k\{x_k\}, \min_\ell\{y_\ell\})$.

4 Linear preferences and spheres

4.1 A fundamental relationship

The normalized gradient field of a linear preference over $A \subset \mathbb{R}^{\ell+1}$ (we assume that $\ell \geq 1$) is a constant vector field and can be represented by a single point on the sphere $S^\ell \subset \mathbb{R}^{\ell+1}$ of radius one. Restricting an aggregation rule $G : \mathcal{P}^n \to \mathcal{P}$ which is defined on the whole space $\mathcal{P}$ of (non-satiated and differentiable) preferences to the space S^ℓ of linear preferences, results in a rule $G : (S^\ell)^n \to \mathcal{P}$. Then, fix some alternative $a \in A \subset \mathbb{R}^{\ell+1}$ and define the map

$$F : (S^\ell)^n \longrightarrow S^\ell : \mathbf{p} \mapsto G(\mathbf{p})(a)$$

where $G(\mathbf{p})(a)$ is the value of the social preference (i.e. the aggregated gradient field) at the point a, and since gradient fields are normalized, the image $G(\mathbf{p})(a)$ belongs to S^ℓ. Hence, the behaviour of the rule F on the sphere is clearly connected with the behaviour of the rule G on $\mathcal{P}$. As a consequence, the study of aggregation rules on spheres has been indicative for the general case.

This section explains the notion of the degree of a continuous map between spheres and its use in different setups of the social choice problem. We start with a lemma that gives a sufficient condition for two maps into a sphere being homotopic. The lemma already reveals the specific behaviour of spheres.

4.1.1 Lemma. Let X be a topological space and let $f, g : X \to S^\ell$ be two continuous maps such that for all $x \in X$ we have $f(x) \neq -g(x)$. Then f and g are homotopic.

Proof. The following map is well defined and is a homotopy between f and g:

$$\vartheta(x,t) : X \times I \to S^\ell : (x,t) \mapsto \frac{(1-t)f(x) \ + \ t\,g(x)}{\|(1-t)f(x) \ + \ t\,g(x)\|}.$$

Next, we define the notion of degree of a continuous map $f : S^\ell \to S^\ell$. Intuitively, the degree of f indicates how many times the sphere (the domain) is wrapped around itself (the sphere in the range). For example, the degree of the identity map is equal to one.

Before we introduce the formal definition let us repeat that the group $\pi_\ell(S^\ell) \cong \mathbb{Z}$ is abelian. The operator in $\pi_\ell(S^\ell)$ is therefore denoted by '+', the neutral element (a constant map) by '0', and a generator (see example 3.2.3 *ii*) by '1'.

4.1.2 Definition. The **degree** of a continuous map $f : S^\ell \to S^\ell$ is the unique integer $\deg(f)$ that satisfies the equation

$$f_*(1) = \deg(f) \times 1,$$

where f_* is the induced homomorphism on the ℓth homotopy group.

4.1.3 Properties.

- If the map $f : S^\ell \to S^\ell$ is not onto then the degree of f is equal to zero,
- The degree of a map is a homotopy invariant (e.g. Dugundji, 1989, Thm 7 4): two maps $f, g : S^\ell \to S^\ell$ are homotopic if and only if $\deg(f) = \deg(g)$.

Let us now return to the aggregation problem. Let $F : S^\ell \times \ldots \times S^\ell \to S^\ell$ be a continuous map. A vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of points on the sphere is said to be a profile. The vector $\mathbf{p}_{-k} = (p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n)$ collects the preferences of individual k's opponents. The notation $(\mathbf{p}_{-k}, p_k)$ is an alternative for $\mathbf{p}$.

Let $\mathbf{p}$ be some fixed profile. One can define a continuous embedding $i_k : S^\ell \longrightarrow (S^\ell)^n : p \mapsto (\mathbf{p}_{-k}, p)$ for each individual $k \in N$. These embeddings induce continuous maps between spheres:

$$F \circ i_k : S^\ell \longrightarrow S^\ell : p \mapsto F(\mathbf{p}_{-k}, p)$$

for $k \in N$. For different profiles $\mathbf{p}$ and $\mathbf{p}'$ the maps $p \mapsto F(\mathbf{p}_{-k}, p)$ and $p \mapsto F(\mathbf{p}'_{-k}, p)$ are homotopic and have therefore the same degree. Hence, we can define the **degree of individual** k as the degree of the map $F \circ i_k$ and this without explicit reference to the vector $\mathbf{p}$.

Note that in case the degree of individual k is different from zero, the map $p \mapsto F(\mathbf{p}_{-k}, p)$

is onto, i.e. $\mathrm{Im}(F \circ i_k) = S^\ell$. For any given vector of preferences of her opponents, such an individual is able to generate any outcome by choosing a suitable preference for herself.

By means of the diagonal embedding $\Delta : p \mapsto (p, p, \ldots, p)$ another map between spheres can be defined:

$$F \circ \Delta : S^\ell \longrightarrow S^\ell : p \mapsto F(p, p, \ldots, p).$$

Now we are able to prove a relationship between the individual degrees and the degree of this diagonal map:

4.1.4 Lemma. Let $F : (S^\ell)^n \to S^\ell$ be a continuous map and let $\Delta, i_1, i_2, \ldots, i_n$ be as above. Then

$$\deg(F \circ \Delta) \;=\; \deg(1) \,+\, \deg(2) \,+ \ldots +\, \deg(n), \tag{2}$$

where $\deg(k)$ is the degree of individual $k \in N$.

Proof. Consider the induced maps at the homotopy level and observe that

$$\begin{cases} \Delta_*(1) = (1, 1, \ldots, 1) \\ (i_k)_*(1) = (0, \ldots, 0, 1, 0, \ldots, 0) \text{ with the 1 at the } k\text{th place.} \end{cases}$$

It follows that $\Delta_*(1) = (i_1)_*(1) + (i_2)_*(1) + \ldots + (i_n)_*(1)$. Apply the group homomorphism F_* and conclude the proof.

For an alternative proof, based upon integration along the sphere, we refer to Aumann (1943) and Lauwers (1999a). The following subsections are devoted to the implications of the above fundamental relationship.

4.2 The topological social choice paradox

Chichilnisky (1979,1980,1982a) questions the existence of a topological aggregator in case the space $\mathcal{P}$ of preferences coincides with a sphere S^ℓ. From the resolution theorem we already know that the answer is negative: spheres are not contractible. An alternative proof goes as follows.

4.2.1 Theorem. There does not exist a map $F : (S^\ell)^n \to S^\ell$ that combines continuity, anonymity, and unanimity.

Proof. Suppose there exists a Chichilnisky rule F on the sphere. Anonymity implies that the individuals all have the same degree: $\deg(i) = \deg(1)$ for $i = 1, 2, \ldots, n$. Unanimity implies that $F \circ \Delta(p) = p$ for all $p \in S^\ell$. Hence, $F \circ \Delta$ is the identity map and has degree equal to one. Apply the fundamental relationship (2) and obtain a contradiction: $1 = n \deg(1)$.

4.2.2 Remark. The special case with two individuals and a two dimensional choice space is equivalent to (*i*) Brouwer's (two dimensional) fixed point theorem (Chichilnisky, 1979) and (*ii*) a particular property of the Möbius strip (Candeal and Induráin, 1994a).

4.2.3 Remark. The present setup excludes **vanishing gradient fields**. Concerning this phenomenon, we insert some results of Chichilnisky (1982a) and Le Breton and Uriarte

(1990a).

Chichilnisky proves that (in the framework of section 2) her results extend to the set $\mathcal{P}_0$ of gradient fields that might vanish on a subset of A of measure zero. Hereby, the set $\mathcal{P}_0$ is equipped with a topology similar to the original setup. Then, a Chichilnisky rule on $\mathcal{P}_0$ is restricted to the set of linear preferences and applied upon one alternative a. This results in a continuous (with respect to the Euclidean topology) map

$$\varphi_0 : (S^\ell)^n \longrightarrow S^\ell \cup \{0\} : \mathbf{p} \mapsto F(\mathbf{p})(a)$$

(the image set is the union of the unit sphere and the origin) that satisfies unanimity and anonymity. Since φ is continuous and $(S^\ell)^n$ is connected, the image of φ is connected and coincides (use unanimity) with S^ℓ. Now theorem 4.2.1 applies.[17]

Le Breton and Uriarte observe that the existence of the above rule φ_0 crucially depends upon the topology the set $S^\ell \cup \{0\}$ is equipped with. For a particular non-Hausdorff topology[18] the map

$$\psi : (S^\ell)^n \longrightarrow S^\ell \cup \{0\} : \mathbf{p} \mapsto \begin{cases} (p_1 + \ldots + p_n)/\|p_1 + \ldots + p_n\| & \text{if } p_1 + \ldots + p_n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is continuous, anonymous, and unanimous. Furthermore, they show that the topology on $\mathcal{P}_0$ considered by Chichilnisky also violates the Hausdorff separation axiom.

4.3 Pareto rules and homotopic dictators

This section confronts the Pareto axiom (as it appears in Arrow's theorem) with the topological approach to social choice. Recall that an aggregation rule satisfies the **Pareto condition** when the following holds for an arbitrary profile of individual preferences and for an arbitrary pair $a, b \in A$ of alternatives: if every individual ranks a higher than b, then so does the social preference. The Pareto condition implies respect of unanimity.

An aggregation rule is **dictatorial** (and individual k is said to be a dictator) if for an arbitrary profile the social preference coincides with the preference of individual k. Such an aggregation rule is a projection:

$$\Pi_k : (S^\ell)^n \longrightarrow S^\ell : \mathbf{p} = (p_1, p_2, \ldots, p_n) \mapsto p_k.$$

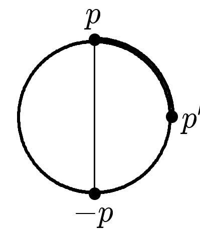
Notice that a dictatorial rule satisfies the Pareto condition. The next lemma is a first step in obtaining a proof of the reverse statement.

4.3.1 Lemma. Let $F : (S^\ell)^n \to S^\ell$ be a continuous map that satisfies the Pareto condition,

[17] Chichilnisky (1985) uses a similar connectedness argument to show that the space of von Neumann-Morgenstern utility functions over a finite set of choices is not contractible and does not admit topological aggregators.

[18] With respect to this particular topology full indifference cannot be distinguished from the rest of the preferences. Or, the only open neighbourhood of 0 is the whole space $S^\ell \cup \{0\}$.

let $\overline{\mathbf{p}}$ be the constant profile $(p, p, \ldots, p)$, and let $k \in N$. Then the aggregated preference $q = F(\,\overline{\mathbf{p}}_{-k}\,, -p)$ belongs to the pair $\{p, -p\}$.



Proof. The Pareto condition implies that the aggregated preference $q' = F(\,\overline{\mathbf{p}}_{-k}\,, p')$ with $p' \neq -p$ is contained in the circular convex hull of p and p'. This is the smallest arch on the great circle through $p, -p$ and p' that contains p and p' (indicated in bold in the figure). Now let the point p' approach $-p$ then continuity implies that q belongs to one particular half ('a' convex hull of p and $-p$) of the great circle through p and $-p$. The same argument applies to the other half of the great circle. Hence the aggregate belongs to the intersection of both halves. This is exactly the pair $\{-p, p\}$.

The combination of lemmas 4.1.1 and 4.3.1 results in the topological equivalence of the Pareto condition and the existence of a dictator in the special case of two individuals.

4.3.2 Corollary. Let $F : S^\ell \times S^\ell \to S^\ell$ be a continuous map that satisfies the Pareto condition. Then F is homotopic to a dictatorial rule.

Proof. Continuity and the Pareto condition imply that

$$\forall p \in S^\ell \text{ we have } F(p, -p) = p \qquad \text{or else} \qquad \forall p \in S^\ell \text{ we have } F(p, -p) = -p.$$

In the first case it follows that $\forall p_1, p_2 : F(p_1, p_2) \neq -p_1$ and in the second case that $\forall p_1, p_2 : F(p_1, p_2) \neq -p_2$. Apply lemma 4.1.1 and conclude the proof.

The following corollary will help us in moving towards the general result.

4.3.3 Corollary. Let $F : (S^\ell)^n \to S^\ell$ be a continuous map that satisfies the Pareto condition. Then (i) the degree of an individual is either one or zero, and (ii) there is exactly one individual with degree equal to one.

Proof. (i) Recall that the degree of individual k does not depend upon the preferences of her opponents. Hence let us determine the degree of the map $F \circ i_k : p' \mapsto F(\overline{\mathbf{p}}_{-k}, p')$ where $\mathbf{p}$ is equal to the constant profile $(p, p, \ldots, p)$. According to the previous lemma we have that $F(\overline{\mathbf{p}}_{-k}, -p)$ is either $-p$ or p. In the former case it follows that (use Pareto)

$$\forall p' \text{ we have } F(\overline{\mathbf{p}}_{-k}, p') \neq -p'$$

which implies that $F \circ i_k$ is homotopic to the identity map (lemma 4.1.1). Hence, the degree of k is equal to one. In the latter case it follows that

$$\forall p' \text{ we have } F(\overline{\mathbf{p}}_{-k}, p') \neq -p.$$

In this case the map $F \circ i_k$ is homotopic to the constant map $p' \mapsto p$ (lemma 4.1.1) and the degree of k equal to zero.
(ii) Since the Pareto condition implies unanimity, the degree of the diagonal $F \circ \Delta$ is equal to one. The fundamental relationship (2) indicates that exactly one individual has a degree equal to one.

Now we are almost ready to prove the topological equivalence between the Pareto condition and dictatorship in the case of an arbitrary number of individuals. First, we state an extra axiom that will be used in the argumentation.

4.3.4 Definition. An aggregation rule F satisfies the **weak positive association condition** (*WPAC*) if $F(\mathbf{p}) = -p_k$ for some k and some profile $\mathbf{p}$, implies that

$$F(\overline{\mathbf{p}}_{-k}, p_k) \neq p_k \quad \text{with } \overline{\mathbf{p}} = (-p_k, -p_k, \ldots, -p_k).$$

The interpretation is clear. If for a particular profile the social outcome is the exact opposite $-p_k$ of the preferences of individual k, then a move of the preferences of the opponents of individual k towards $-p_k$ cannot change the outcome into p_k. Obviously, a dictatorial rule does satisfy *WPAC*.

4.3.5 Theorem (Chichilnisky, 1982c)**.** Let $F : (S^\ell)^n \to S^\ell$ be a continuous map that satisfies the Pareto condition and *WPAC*. Then F is homotopic to a projection $\Pi_j : \mathbf{p} \mapsto p_j$ for a unique $j \in \{1, 2, \ldots, n\}$.

Proof. From corollary 4.3.3 we know that exactly one individual has a degree equal to one, and that the others have a degree equal to zero. Consider for each individual $k \in N$ the following condition:

$$\forall \mathbf{p} \in (S^\ell)^n \text{ we have } F(\mathbf{p}) \neq -p_k \ = -\Pi_k(\mathbf{p}). \tag{2}$$

Observe that

- if for individual k condition (2) holds, then the maps F and Π_k are homotopic (use lemma 4.1.1),
- if for individual k condition (2) does not hold, then the degree of k is equal to zero.

We clarify the second assertion. Let $F(\mathbf{p}) = -p_k$ for some profile $\mathbf{p}$. *WPAC* implies that for the particular constant profile $\overline{\mathbf{p}} = (-p_k, -p_k, \ldots, -p_k)$ the map

$$F \circ i_k : (S^\ell)^n \longrightarrow S^\ell : p \mapsto F(\overline{\mathbf{p}}_{-k}, p)$$

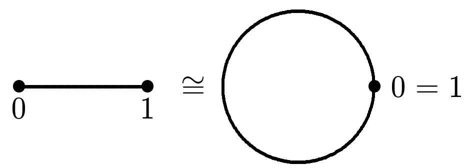
is not onto (p_k does not belong to the image of $F \circ i_k$). Hence the degree of individual k is equal to zero.
Apparently condition (2) holds for exactly one individual. This completes the proof.

4.3.6 Remark. Koshevoy (1997) shows that in the two dimensional case the *WPAC* can be dropped: a continuous map $F : (S^1)^n \to S^1$ that satisfies the Pareto condition is homotopic to a dictatorial rule.

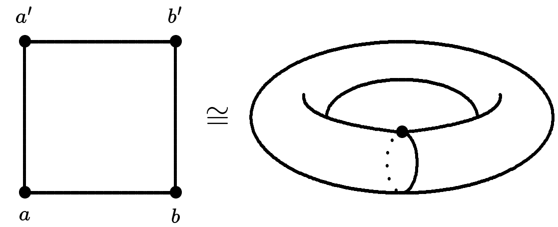
4.3.7 Remark. Chichilnisky's theorem states that an aggregation rule that satisfies the Pareto condition and *WPAC* can be continuously deformed into a dictatorial rule. Again, as such homotopies might be 'wild', the interpretation of this statement is not obvious.

Saari (1997) uses the notion of 'dominant' voters to interpret this theorem. Such an individual plays a dominant (but not total) role in determining the outcome. Let us

illustrate his approach in the two person two dimensional case $S^1 \times S^1 \to S^1$.
The circle S^1 can be seen as the interval I the two endpoints of which are identified. Similarly, the torus $S^1 \times S^1$ results from a square $I \times I$ after identifying the lines $[a, b]$ with $[a', b']$ and $[a, a']$ with $[b, b']$.

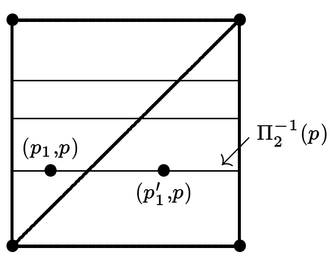


The circle S^1

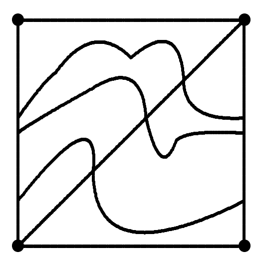


The torus $S^1 \times S^1$

Now we consider two aggregation rules, the dictatorial rule $\Pi_2 : (p_1, p_2) \mapsto p_2$ and a Pareto rule F which is homotopic to Π_2. The next picture shows some of the level sets for both maps. The domain $S^1 \times S^1$ is represented by the square, the edges of which have to be identified (as described above). Observe that, because both rules satisfy the Pareto condition, they also satisfy the weaker unanimity condition. Therefore, each level set intersects the diagonal exactly once.
Consider first the dictatorial map. The level sets $\Pi_2^{-1}(p) = \{ (p_1, p_2) \,|\, \Pi_2(p_1, p_2) = p \}$ are horizontal lines. Indeed, as the preferences of individual 1 do not matter, the profiles (p_1, p) and (p'_1, p) belong to the same level set.



Level sets of Π_2



Level sets of F

The second part of the picture deals with a non-dictatorial rule $F : S^1 \times S^1 \to S^1$ that satisfies the Pareto condition. As the rule F is homotopic to Π_2, the level sets are deformations of the horizontal lines. Note that locally (in particular in the neighbourhood of the diagonal) this rule F gives more 'power' to the first player (the level sets are almost vertical).

The following lemma shows that the concept of 'topological' dictatorship becomes empty in case the space of preferences is contractible.

4.3.8 Lemma. Let Y be a contractible topological space and let X be any topological space. Let $f, g : X \to Y$ be two continuous maps. Then f and g are homotopic.

Proof. We show that any continuous map $f : X \to Y$ is homotopic to a constant map. Since the relation 'homotopic to' is an equivalence relation the lemma follows. The contractibility of Y implies the existence of a homotopy $\vartheta : Y \times I \to Y$ with $\vartheta(y,0) = y$ and $\vartheta(y,1) = y_0$ where $y_0 \in Y$. Use this homotopy ϑ to define a new map:

$$\vartheta' : X \times I \to Y : (x,t) \mapsto \vartheta'(x,t) = \vartheta(f(x),t).$$

The map ϑ' establishes a homotopy between $f(\,.\,) = \vartheta'(\,.\,,0)$ and $C_{y_0} = \vartheta'(\,.\,,1)$.

If the topological space $\mathcal{P}$ of preferences is contractible, an 'anonymous' aggregation rule exists (use the resolution theorem) but such a rule is homotopic to a 'dictatorial' rule. In addition, all dictatorial rules are homotopic to each other! In other words, the concept of topological dictator is only significant in case the space of preferences is not contractible.

This subsection ends with the statement of two results by Baryshnikov (1994). He narrows the concept of homotopy and refines the analysis of Chichilnisky: let $\mathcal{P}$ be a space of preferences and let $f, g : \mathcal{P}^n \to \mathcal{P}$ be two continuous maps that satisfy the Pareto condition. Then f and g are said to be **Pareto-isotopic** if there exists a homotopy ϑ between f and g such that the intermediate maps $\vartheta(\,.\,,t)$ satisfy the Pareto condition for all $t \in I$.[19] Equipped with this new concept, he establishes the following results:

4.3.9 Theorem. Let $F : (S^\ell)^n \to S^\ell$ be a continuous map that satisfies the Pareto condition and *WPAC*. Then F is Pareto-isotopic to a projection $\Pi_j : \mathbf{p} \mapsto p_j$ for a unique $j \in \{1, 2, \ldots, n\}$.

4.3.10 Theorem. There exists a continuous map $F : (S^2)^4 \to S^2$ that satisfies the Pareto condition and that is not Pareto-isotopic to a dictatorial rule.

Baryshnikov interprets this second result as a possibility theorem. It shows the existence of a Pareto rule which cannot be deformed into a dictatorial rule unless one moves away from the Pareto condition during the continuous deformation.

4.4 Variations on the theme: no-veto and manipulation

This subsection shows that without the introduction of 'homotopic dictators' some interesting results were obtained in the framework of continuous aggregation maps on the sphere. The results we have in mind involve the notion of 'no-veto' and of 'manipulation'. In view of the following list of properties, the proofs are straightforward.

- $\deg(F \circ \Delta) \;=\; \deg(1) \,+\, \deg(2) \,+ \ldots +\, \deg(n)\,,$ (2)
- if F is unanimous, then $\deg(F \circ \Delta) = 1\,,$
- if F is Pareto, then $\deg(i) \in \{0, 1\}$ and $\deg(F \circ \Delta) = 1$.

Let us now repeat the definition of the 'no-veto condition':

4.4.1 Definition. A rule F satisfies the **no-veto** condition if for all constant profiles

[19] Replacing the Pareto axiom with a different (set of) axiom(s) results in a different concept of isotopy.

$\overline{\mathbf{p}} = (p, p, \ldots, p)$ we have $F(\overline{\mathbf{p}}_{-k}, -p) \neq -p$.

This axiom states that if one individual holds completely opposite preferences from the others, then the aggregate will not agree with this single individual.

4.4.2 Theorem (Chichilnisky, 1982b; Mehta, 1997)**.** There does not exist a continuous map $F : (S^\ell)^n \to S^\ell$ (with $n \geq 3$) that satisfies the Pareto condition and the no-veto condition.

Proof. The no-veto condition implies that the degree of each individual is equal to zero. Indeed, consider the embedding $i_k : p' \mapsto (\overline{\mathbf{p}}_{-k}, p')$. The image of the composite map $F \circ i_k$ does not contain the point $-p$ (use the Pareto condition). Hence $\deg(k) = 0$ for all k. This is in contradiction to (2).

A reformulation of this theorem reads as follows: an aggregation procedure that combines the Pareto and the no-veto conditions must be instable (in the sense of non-continuous).[20]

Secondly, we present the notion of 'manipulator':

4.4.3 Definition. An individual k is said to be a **manipulator** if for all $p \in S^\ell$ and for all profiles $\mathbf{p} \in (S^\ell)^n$ there exists a p' such that $F(\mathbf{p}_{-k}, p') = p$.

Obviously dictators are manipulators. Also, rules homotopic to a dictatorial one induce a unique manipulator. The converse of the previous statement is not true: the existence of a manipulator does not imply that a rule can be continuously deformed into a dictatorial rule. Furthermore, an individual with degree different from zero is a manipulator.

4.4.4 Theorem (Chichilnisky, 1983,1993a)**.** Let the aggregation map F be continuous and unanimous. Then there exists at least one manipulator.

In Chichilnisky and Heal (1984) the unanimity condition in the previous theorem is weakened to 'citizen's sovereignty'. This condition states that $F \circ \Delta$ is of nonzero degree, and implies that by coordinating their strategies the individuals can achieve all outcomes.

Replace in the previous theorem unanimity with the stronger Pareto condition, and obtain:

4.4.5 Theorem (Chichilnisky, 1983; Baryshnikov, 1994; Koshevoy, 1997)**.** Let the aggregation map F be continuous and satisfy the Pareto condition. Then there exists a unique manipulator.

5 The topological approach to Arrow's theorem

Arrow's impossibility theorem and its proof are of a combinatorial nature. However, Baryshnikov indicated how to transform a discrete model in social choice theory into a simplicial complex equipped with a non-trivial topology. Then, the resulting model can be studied in the above exposed topological framework. Developing this strategy, Baryshnikov proved that Arrow's theorem is implied by Chichilnisky's social choice paradox on spheres. The topological approach is able to unify both impossibility theorems!

[20]Chichilnisky (1982a) and Mehta (1997) use a 'decisive majority axiom' which is stronger than the no-veto axiom. MacIntyre (1998) obtains a result similar to theorem 4.4.2 in case $\ell = 1$.

We want to stress that during this 'topologizing' the Arrowian model itself is not augmented with additional structure. The topological structure is a natural consequence of the reformulation of the model. This section introduces some additional concepts and repeats some of the arguments of Baryshnikov (1993,1997).

5.1 Simplicial complexes, the nerve of a cover

We start with the notion of simplicial complex. A simplex is just a generalization of a triangle or a tetrahedron to higher dimensions. Different simplexes can be fitted together to obtain a space called a polyhedron. This 'fitting together' happens in such a way that two simplexes either do not meet or meet in a common vertex, edge, or face.

5.1.1 Definition. A **q-simplex** σ in $\mathbb{R}^m$ is the set of points $\sum_{i=0}^{q} \lambda_i a_i$ where $a_0, a_1, \ldots, a_q$ are $q+1$ points in $\mathbb{R}^m$ that are independent (i.e. the vectors $a_1 - a_0, a_2 - a_0, \ldots, a_q - a_0$ are linearly independent) and the λ_i are non-negative real numbers adding up to 1. The q-simplex is the convex hull of the $q+1$ points $a_0, a_1, \ldots, a_q$.
The points a_i are called **vertices** of σ and are said to span the simplex. The simplex σ' spanned by a subset $\{a_{i_0}, a_{i_1}, \ldots, a_{i_p}\}$ of the vertices of σ is said to be a **face** of σ.
A (finite) **simplicial complex** K is a finite set of simplexes, all contained in some Euclidean space $\mathbb{R}^m$, such that

- if σ is a simplex of K and σ' is a face of σ, then σ' is in K
- if σ and τ are simplexes of K, then $\sigma \cap \tau$ either is empty or is a common face of σ and τ.

Such a simplicial complex K is embedded in $\mathbb{R}^m$ and usually inherits the Euclidean topology, in which case K is also called a polyhedron.

A **map** $f : K \to L$ between two simplicial complexes is said to be **simplicial** if it has the following properties:

- If a is a vertex of K, then $f(a)$ is a vertex of L,
- If σ is a simplex of K, then $f(\sigma)$ is a simplex of L,
- f is linear on each simplex, i.e. if $x = \Sigma\, \lambda_i a_i$ is in a simplex σ spanned by the vertices $a_1, a_2 \ldots, a_l$, then $f(x) = \Sigma\, \lambda_i f(a_i)$.

Simplicial maps are continuous with respect to the Euclidean topology. Also, a simplicial map $f : K \to L$ is completely defined by its values on the vertices of K.

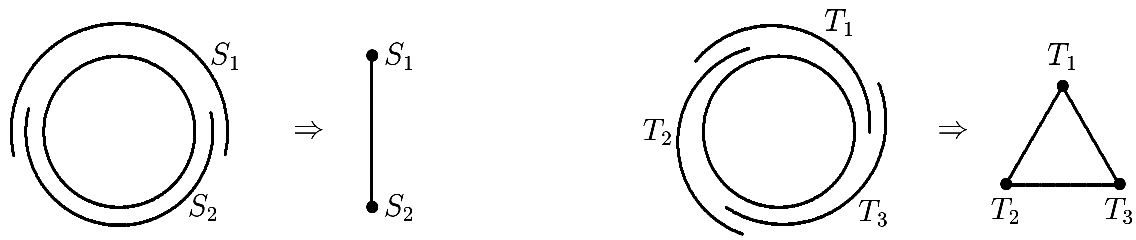
One of methods to 'topologize' a certain object makes use of covers and nerves. The definitions go as follows (e.g. Eilenberg and Steenrod, 1952):

5.1.2 Definition. Let S be a set, and let $S_1, S_2, \ldots, S_t$ be non-empty subsets of S that **cover** S, i.e.

$$S \subset S_1 \cup S_2 \cup \ldots \cup S_t.$$

The **nerve** of S with respect to this cover is a simplicial complex the vertices of which are t points $S_1, S_2, \ldots, S_t$. The nerve of S contains precisely those simplexes spanned by $\{ S_{i_1}, S_{i_2}, \ldots, S_{i_{t'}} \}$ for which the intersection $S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{t'}}$ is non-empty.

5.1.3 Example. In order to illustrate these concepts, let the set S be a circle. We provide two situations. In the first case the circle is covered by two sets S_1 and S_2. The nerve with respect to this cover is a straight line between the points S_1 and S_2. In the second case, the cover consists out of three subsets T_1, T_2 and T_3. The nerve is now a complex with three line segments. The simplex spanned by $\{ T_1, T_2, T_3 \}$ does not belong to the nerve: the three sets have an empty intersection.



Two covers of the circle and their nerves.

In case the original set S is equipped with a topology the original topology can be 'compared' with the topology of the nerve. The previous picture illustrates this point. In the first situation, the nerve is a line segment (therefore contractible) and is different from (read 'not homotopic to') the original space. In the second situation, the nerve coincides with the border of a triangle and is homotopic to the original circle (with the Euclidean topology).

The fundamental principle behind this phenomenon is the so-called nerve-theorem: roughly speaking, if the intersection of any subset of the cover is either empty (i.c. $T_1 \cap T_2 \cap T_3 = \emptyset$) or contractible (i.c. $T_i \cap T_j$), then the nerve is homotopic to the original space.

5.2 Arrow's theorem

Now we return to the Arrowian model. $A = \{a_1, a_2, \ldots, a_k\}$ is a finite set of alternatives and $\mathcal{P}$ is the finite set of strict preference (i.e. complete, transitive, and asymmetric) relations in A. Each individual $i \in N$ is equipped with a preference relation $P_i \in \mathcal{P}$, and $(a_j, a_{j'}) \in P_i$ stands for 'individual i prefers alternative a_j above $a_{j'}$'. An Arrowian aggregation rule H is a map

$$H : \mathcal{P}^n \longrightarrow \mathcal{P} : \mathbf{P} = (P_1, P_2, \ldots, P_n) \mapsto H(\mathbf{P})$$

that satisfies

- the weak Pareto condition, i.e. if for two alternatives $a_j, a_{j'}$ we have $(a_j, a_{j'}) \in P_i$ for all individuals $i \in N$, then $(a_j, a_{j'}) \in H(\mathbf{P})$,

- Arrow's independence axiom, i.e. if the restrictions of two profiles $\mathbf{P}, \mathbf{P}' \in \mathcal{P}^n$ to a pair $a_j, a_{j'}$ of alternatives coincide then the restrictions of the aggregate preferences $H(\mathbf{P})$ and $H(\mathbf{P}')$ to this pair also coincide.

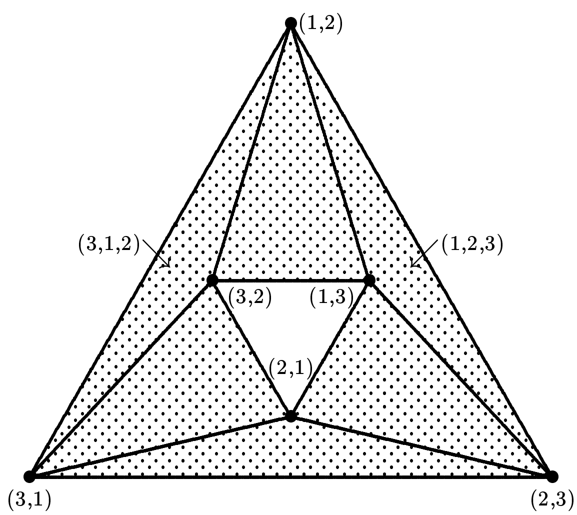
Next, we look for an appropriate cover and its nerve.

- Denote by (i, j) the subset of $\mathcal{P}$ of preferences for which alternative a_i is ranked above a_j for $i, j = 1, 2, \ldots, k$, $i \neq j$. These $k(k-1)$ subsets form a cover of $\mathcal{P}$, i.e.

$$\mathcal{P} \subset (1,2) \cup (1,3) \cup \ldots \cup (k, k-1).$$

- The subsets (i, j) become the vertices of the nerve $\mathcal{N}(\mathcal{P})$.
- The straight line between two points (i, j) and (k, l) belongs to the complex $\mathcal{N}(\mathcal{P})$ if and only if the two sets (i, j) and (k, l) in $\mathcal{P}$ have a non-empty intersection.
  The area (volume,..) between three (four,..) points belongs to the complex $\mathcal{N}(\mathcal{P})$ if and only if the corresponding sets in $\mathcal{P}$ have a non-empty intersection.

5.2.1. The following picture illustrates the case of three alternatives.



Let $A = \{a_1, a_2, a_3\}$. The cover of $\mathcal{P}$ consists of the six sets (i, j). Each set becomes a point in the nerve, and is connected with the four points different from (j, i). The shaded areas spanned by three neighbouring points refer to complete rankings, e.g. $(1, 2, 3)$ refers to the order that ranks a_1 at the top and a_3 at the bottom. The number of such areas is equal to $3! = 6$. Observe that the area spanned by $(1, 3)$, $(2, 1)$, $(3, 2)$ (and also by $(1, 2)$, $(2, 3)$, $(3, 1)$) is not shaded: the three sets of preferences these points refer to have an empty intersection. The space $\mathcal{N}(\mathcal{P})$ in $\mathbb{R}^2$ is homotopic to the circle S^1.

A generalization of this picture leads to the following theorem:

5.2.2 Theorem. Let $\mathcal{P}$ be the set of preferences over a set of k alternatives. Then, the above defined simplicial complex $\mathcal{N}(\mathcal{P})$ is homotopic to the $(k-2)$-dimensional sphere S^{k-2}.

Prior to the proof, we provide some further insight in the above construction: the complex $\mathcal{N}(\mathcal{P})$ is of dimension $(k+1)(k-2)/2$. Indeed, a face cannot be spanned by a set of $(k+1)(k-2)/2 + 2$ vertices. Such a set of vertices will contain the points (i, j) and (j, i) for at least one pair i, j of alternatives, which implies that the simplex spanned by these vertices does not belong to the complex. Furthermore, some faces of $\mathcal{N}(\mathcal{P})$ are spanned by a $(k+1)(k-2)/2+1$-tuple of vertices (a complete order as the intersection of $k(k-1)/2$ sets of the form (i,j)). The dimension of such a simplex is equal to $(k+1)(k-2)/2$. Since any sequence $(i_1,j_1),(i_2,j_2),\dots$ with a nonempty intersection can be extended to a sequence of length equal to $k(k-1)/2$, the complex $\mathcal{N}(\mathcal{P})$ is the union of such faces of maximal dimension. As we consider $k!$ strict orderings, there are $k!$ maximal-dimensional simplexes.

Proof of 5.2.2. Consider the space $\mathcal{P}^* = I^k - \{(x,x,\dots,x)\,|\,x\in I\}$. This space is homotopic to the sphere $S^{k-2}\subset I\!R^{k-1}$, and is covered by the subsets

$$(i,j)^* = \{\mathbf{x}\in I^k\,|\,x_i > x_j\},$$

for $i\neq j\in\{1,2,\dots,k\}$. The nerve $\mathcal{N}(\mathcal{P}^*)$ of this cover of $\mathcal{P}^*$ is homeomorphic to the simplicial complex $\mathcal{N}(\mathcal{P})$. To explain this, we start from the map $(i,j)\mapsto(i,j)^*$ defined between the two sets of vertices. The intersection of a subset of the cover of $\mathcal{P}$ is empty if and only if the intersection of the corresponding subset of the cover of $\mathcal{P}^*$ is empty. It follows that this map linearly extends to a homeomorphism

$$\mathcal{N}(\mathcal{P})\longrightarrow\mathcal{N}(\mathcal{P}^*):\Sigma\,\lambda_{i,j}(i,j)\mapsto\Sigma\,\lambda_{i,j}(i,j)^*.$$

We conclude the proof by showing that $\mathcal{N}(\mathcal{P}^*)$ is homotopic to $\mathcal{P}^*$. This statement follows from the nerve-theorem: the intersection of any collection of covering sets is either empty or contractible.

Now, we reflect on the aggregation map. In order to make use of the topological approach this map should be continuous or even simplicial. Arrow's independence axiom in combination with this need for continuity will guide us in defining a simplicial complex associated with the space $\mathcal{P}^n$ of profiles. Consider the following subsets of $\mathcal{P}^n$:

$$(i,j)^{\mathrm{s}} = (i,j)^{(s_1,s_2,\dots,s_n)} = \{\,\mathbf{P}\,|P_\ell\in(i,j) \text{ if } s_\ell=+ \text{ and } P_\ell\in(j,i) \text{ if } s_\ell=-\},$$

for all $i<j=1,2,\dots,k$ and all vectors $\mathbf{s}=(s_1,s_2,\dots,s_n)\in\{+,-\}^n$. These subsets cover the space $\mathcal{P}^n$ of profiles. The independence axiom implies that such a subset $(i,j)^{\mathrm{s}}$ is aggregated into one vertex of $\mathcal{N}(\mathcal{P})$ (either (i,j) or (j,i)). Hence, it seems natural to take the nerve $\mathcal{N}(\mathcal{P}^n)$ of this cover as the simplicial complex associated with the space of profiles. Then, an Arrow aggregation rule H maps a vertex of $\mathcal{N}(\mathcal{P}^n)$ onto a vertex of $\mathcal{N}(\mathcal{P})$ and extends to a simplicial map

$$H:\mathcal{N}(\mathcal{P}^n)\longrightarrow\mathcal{N}(\mathcal{P}).$$

Indeed, if some vertices in $\mathcal{N}(\mathcal{P}^n)$ span a simplex, then there images in $\mathcal{N}(\mathcal{P})$ also do. Note that $\mathcal{N}(\mathcal{P}^n)$ differs from $(\mathcal{N}(\mathcal{P}))^n$. However, the lower homotopy groups behave well:

5.2.3 Theorem. Let $\mathcal{P}$ be the set of strict preferences over a set of k alternatives. Then, the homotopy groups $\pi_\ell(\mathcal{N}(\mathcal{P}^n))$ of the simplicial complex $\mathcal{N}(\mathcal{P}^n)$ are trivial for $1\leq\ell<k-2$ and $\pi_{k-2}(\mathcal{N}(\mathcal{P}^n))$ is isomorphic to $\mathbb{Z}^n$.

As a consequence, the following homomorphisms at the homotopy level come up again:

$$\begin{cases} \Delta_* : \mathbb{Z} \longrightarrow \mathbb{Z}^n : k \mapsto (k, k, \ldots, k), \\ i_{j*} : \mathbb{Z} \longrightarrow \mathbb{Z}^n : k \mapsto (0, \ldots, 0, k, 0, \ldots, 0), \ \text{ with } j = 1, 2, \ldots, n, \\ H_* : \mathbb{Z}^n \longrightarrow \mathbb{Z} : \mathbf{k} = (k_1, k_2, \ldots, k_n) \mapsto H_*(\mathbf{k}). \end{cases}$$

Hereby, is $\Delta : \mathcal{N}(\mathcal{P}) \to \mathcal{N}(\mathcal{P}^n)$ defined by $(i, j) \mapsto (i, j)^{(+,+,\ldots,+)}$. The Pareto axiom implies that $H \circ \Delta$ is the identity map on $\mathcal{N}(\mathcal{P})$. Since H_* is a homomorphism it follows that

$$H_*(\mathbf{k}) = a_1\, k_1 \ + \ a_2\, k_2 \ + \ldots + \ a_n\, k_n, \quad \text{with } a_j \in \mathbb{Z}. \tag{3}$$

The Pareto condition implies that $a_1 + a_2 + \ldots + a_n = 1$ (substitute $k_j = 1$ for all j in equation (3)). Let us return to the three-alternatives-case (5.2.1) to clarify the maps i_{j*}. The first homotopy group $\pi_1(\mathcal{N}(\mathcal{P})) = \mathbb{Z}$ is then easy to visualize: e.g. the shortest closed path through the points $(1, 2), (3, 1)$ and $(2, 3)$, in that order, is a generator[21], and is denoted by '1'. The shortest closed path through $(1, 2), (3, 1)$ and $(3, 2)$ can be continuously deformed to a constant path, and is denoted by '0'. The embedding $i_{j*}(1)$ in $\mathcal{N}(\mathcal{P}^n)$ is now clear: path 1 is mapped into the closed path through

$$(1, 2)^{(+,+,\ldots,+)}, (3, 1)^{(+,+,\ldots,+)} \quad \text{and} \ \ (2, 3)^{(-,\ldots,-,+,-,\ldots,-)},$$

with in the final vector the $+$ at the jth position. The projection of $i_{j*}(1)$ to the jth (resp. another) component is equal to 1 (resp. 0). And, the image of $i_{j*}(1)$ under the aggregation rule H is the closed path through

$$\text{either} \ \ (1, 2), (3, 1), (2, 3) \quad \text{or} \ \ (1, 2), (3, 1), (3, 2).$$

In the first case individual j is a dictator and has degree equal to 1. In the second case the degree of individual j is equal to 0. This observation extends to the general case:

5.2.4 Proposition. If individual j is a dictator, then a_j in equation (3) is equal to 1, otherwise it is 0.

Arrow's theorem follows from the equality $\Sigma_1^n\, a_j = 1$: dictatorial rules are the only aggregation rules that combine binary independence and Pareto.

Concerning the robustness of this link between Arrow's and Chichilnisky's framework, Baryshnikov indicates how his approach applies when the universal domain condition is replaced with the free-triple assumption.

We close this section with a remarkable observation by Baryshnikov. Look again at the picture in (5.2.1). The complex $\mathcal{N}(\mathcal{P})$ is homotopic to a circle, is therefore not contractible, and does not admit aggregation rules. Now suppose that two neighbouring preference orders, e.g. $(3, 1, 2)$ and $(1, 3, 2)$ are inaccessible for the individuals. In other words, the two corresponding triangles in the simplex are wiped out. Then, the remaining part (a simplex built up with four triangles) is contractible and allows for Arrow aggregators. Indeed, the remaining simplex corresponds with the four single peaked preferences along the tree $1 < 2 < 3$!

[21] To make the exposition concrete let us fix the base point x_0 at $(1, 2)$.

6 A short survey

This section provides further bibliographic notes and lists some of the results obtained in frameworks similar to the Chichilnisky model.

6.1 Discrete models

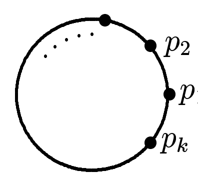
Several authors have developed discrete versions of the Chichilnisky model. In such models the continuity demand is reformulated in terms of a proximity preservation condition: the aggregation rule is relatively insensitive to small changes in the individual preferences.

6.1.1. Baigent (1987) considers an Arrowian model: the set A of alternatives is finite and has no further structure, the set of preferences on A is denoted by $\mathcal{P}$. Besides aggregation rules or social welfare functions, Baigent also deals with social choice functions (i.e. mappings from the set of profiles to the collection $\mathcal{D}(A)$ of subsets of A). He considers the following 'continuity' condition:

An aggregation rule $F : \mathcal{P}^n \to \mathcal{P}$; resp. a social choice function $f : \mathcal{P}^n \to \mathcal{D}(A)$; preserves proximity if there exists any metric[22] δ on $\mathcal{P}$; resp. any metric μ on $\mathcal{D}(A)$; such that for all profiles $\mathbf{P}, \mathbf{P}'$ and $\mathbf{P}''$ we have

$$\Sigma_{i=1}^n \delta(P_i, P_i') < \Sigma_{i=1}^n \delta(P_i, P_i'') \quad \Rightarrow \quad \begin{cases} \delta\left(F(\mathbf{P}), F(\mathbf{P}')\right) \, < \, \delta\left(F(\mathbf{P}), F(\mathbf{P}'')\right); \\ \text{resp. } \mu\left(f(\mathbf{P}), f(\mathbf{P}')\right) \, < \, \mu\left(f(\mathbf{P}), f(\mathbf{P}'')\right). \end{cases}$$

Apparently each proximity preservation condition is incompatible with anonymity and unanimity. In subsequent work, however, these proximity conditions are relaxed and positive results are obtained (Baigent, 1989; Nitzan, 1989).



6.1.2. In Baryshnikov (1997) the set A of alternatives is more structured. He considers the 'beach party problem': a group of n people is choosing a picnic spot on the beach that surrounds a lake. As the beach is every now and then provided with a picnic site the set A of alternatives consists out of k points along a circle S^1. And, each person has her own beloved place to stay. An aggregation rule is a map $F : A^n \to A$ that satisfies unanimity, anonymity, and stability (if one individual changes her opinion and wants to move to a site next to her previous choice, then the output of the aggregation rule changes at most to a site next to the previous output). Again, an impossibility result appears: in case the number of sites is large enough ($k > 2n$) these three axioms are incompatible.

6.2 Strategy-proofness

A game form consists of a set M of messages or strategies, a set A of outcomes, and a map g that associates to each n-tuple of individual messages an outcome in A. By assuming

[22] A natural example of such a metric δ on $\mathcal{P}$ is: $\delta(P, P') = |(P - P') \cup (P' - P)|$ where $P, P' \subset A \times A$.

that the players (i.e. individuals) have preferences over social outcomes one can regard a social decision procedure as a game with the aggregation map as a game form. Formally, the set M of messages coincides with the set $\mathcal{P}$ of preferences over A, and an aggregation rule $F : \mathcal{P}^n \to \mathcal{P}$ induces a game form on $M = \mathcal{P}$ where the outcome is the (set of) alternative(s) that maximizes the aggregate preference.

6.2.1. In this context, an aggregation map is said to be strategy-proof if it a Nash equilibrium that the players report their most preferred outcomes. Assume that the preferences are linear and that the aggregation map is unanimous. Theorem 4.4.4 implies that a strategy-proof game form $g : (S^k)^n \to S^k$ is dictatorial (e.g. Chichilnisky, 1983). Rasmussen (1997) considers a collection of preferences over A with unique maxima such that every outcome in a is the maximum of some preference in the collection. He extends the previous result: a strategy-proof and unanimous map $g : M^2 = A^2 \to A$ is dictatorial as soon A is a path-connected H'-space for which any map from A to A that is homotopic to the identity map on A is onto.

Chichilnisky (1983) and Chichilnisky and Heal (1997a) restrict the space of preferences to single peaked preferences and give a complete characterization of games which induce truthful revelation of the players' preferences as dominant strategies: the only games satisfying this condition are locally constant or dictatorial.

6.2.2. MacIntyre (1998) investigates the two person two dimensional case and shows that the imposition of monotonicity (demanding that when the preferences of individual i move closer to those of the other player j, the social outcome must not penalize player j) turns a continuous Pareto rule on the circle into a dictatorial rule. No algebraic topology is used in his proof.

6.3 Infinite populations

Candeal et al (1992,1997a), Chichilnisky and Heal (1997b), Efimov and Koshevoy (1994), Horvath (1995), Koshevoy (1997), Lauwers (1993,1997) all investigate the limiting behaviour of the Chichilnisky model in case the set of individuals becomes infinitely large.[23]

6.3.1. In extending the model to an infinite framework one has to define infinite versions of the aggregation axioms (i.c. continuity and anonymity). Concentrating on the continuity axiom, it was proven that continuity with respect to the product topology is an extremely strong condition: in combination with a weak anonymity demand it turns a rule into a constant map. With respect to the larger uniform topology more maps are continuous and possibility results turn up. But then the outlook of an infinite rule crucially depends upon the anonymity condition. We mention one result close to the original resolution theorem of Chichilnisky and Heal: A compact and path-connected parafinite CW-complex $\mathcal{P}$ admits an infinite aggregation rule that satisfies bounded anonymity, continuity (with respect to the uniform topology), and unanimity if and only if the space $\mathcal{P}$ is contractible (Lauwers,

[23] In Lauwers (1995) this exercise is done for different aggregation models. Candeal et al (1997b) and Horvath (1995) tackle the continuum population case.

1995,1997). Bounded anonymity is an infinite version of the anonymity condition that imposes identification of, for example, the profiles in which the odd and the even numbered individuals exchanged their preferences.

6.3.2. Also, the manipulation result (theorem 4.4.5) is studied in the infinite population case. Again, the choice of the topology is crucial. Continuity with respect to the product topology implies the existence of a single individual that manipulates the outcome (Koshevoy, 1997). In case the uniform topology is imposed, a result close to the Arrowian approach shows up: the set of manipulating coalitions is an ultrafilter (Lauwers, 1999b).

6.4 Means

The notion of 'aggregation' also occurs in other fields such as, for instance, in group theory (algebraic n-rules) and in the theory of measurement (fusion of information). In both areas the axiomatic approach has been developed.

6.4.1 A generalized algebraic n-mean on a group[24] $G, +$ is a map $F : G^n \to G$ that (i) is unanimous, (ii) satisfies $nF(\mathbf{x}) = x_1 + \ldots + x_n$ for all $\mathbf{x} \in G^n$, and (iii) respects the algebraic structure, i.e. $G(x_1 + y_1, \ldots, x_n + y_n) = G(\mathbf{x}) + G(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in G^n$. The arithmetic mean is one of the basic examples. In case the group G is equipped with a topology, one can define a topological n-mean (i.e. a Chichilnisky rule) on G. In general the topological and the algebraic n-means are quite different. However, for compact connected topological abelian groups the existence of a topological n-mean is equivalent to the existence of an algebraic n-mean (Keesling, 1972). Candeal and Induráin (1994b) provide an excellent survey on the relationships between the existence of different concepts of n-means. They also investigate the implications on the cohomology groups of G. Their approach has led to an alternative proof of the resolution theorem (cf. remark 3.4.1).

6.4.2. Fusion of information is a major item in data analysis. As the available information might be imperfect, e.g. some of its elements are uncertain or imprecise, the continuity condition becomes a major requirement.

The set on which the 'mean' is defined is supposed to be a topological lattice. And the axioms imposed upon an aggregator F are (i) continuity, (ii) symmetry (i.e. anonymity), and (iii) $\min\{x_1, \ldots, x_n\} \leq F(\mathbf{x}) \leq \max\{x_1, \ldots, x_n\}$. Furthermore, in order to take the level of measurement into account, an invariance axiom is imposed. E.g. if the data are ordinal an aggregator F becomes meaningful if (iv) $f(F(\mathbf{x})) = F(f(x_1), \ldots, f(x_n))$ for any continuous strictly increasing function f. Apparently, these axioms already characterize an aggregation rule: the order statistics are the only means that satisfy all four conditions $(i - iv)$. For further results we refer to Marichal and Roubens (1993) and Ovchinnikov (1996,1998).

[24] In order to admit a generalized algebraic mean, the group has to be abelian. Therefore, we write $G, +$.

7 Conclusion

This paper focussed on the use of homotopy theory in the problem of preference-aggregation. As such it exhibits (at least) two shortcomings.

Firstly, the recent results on the link between social choice and general equilibrium are not inserted. For the equivalence of the existence of a competitive equilibrium in an exchange economy and the existence of topological social choice rules, we refer to Chichilnisky (1993b,1997).[25]
Secondly, we want to stress that homotopy theory is only one chapter in the area of algebraic topology. Also homology and its dual theory of cohomology study topological spaces through the extraction of algebraic data. But since the definitions of homology or of cohomology groups are somewhat more involved than the definition of the Hurewicz homotopy groups it seems to be natural to serve up the more accessible homotopy theory first.[26] Furthermore, the classification of spaces on the base of homotopy types is finer than on the base of homology type (i.e. spaces of the same homotopy type generate isomorphic homology groups). From a practical point of view, however, homology groups of CW-complexes are easier to determine: for example, the homotopy group $\pi_n(S^k)$ is not yet known in the general case while the homology group $H_n(S^k;\mathbb{Z})$ is equal to $\mathbb{Z}$ for $n = 0, k$ and to 0 otherwise.

Much of the earlier work in social choice theory mainly considered discrete Arrowian models. As a consequence there was (in contrast to most other parts in economics) little mention of continuity (McManus, 1982; Sen, 1986).[27] The major breakthrough of the continuity condition as an alternative for Arrow's independence axiom is due to Chichilnisky. Besides that, she introduced algebraic topology into social choice. A well established field in mathematics became a servant and generated a bunch of relevant results. The more recent research revealed the possibility to tackle also discrete models by means of algebraic topology. In addition, the connections between aggregation of preferences, general equilibrium, and game theory guarantee a further integration of algebraic topology into the mainstream economics.

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[25] Baigent (1997) provides a short discussion on Chichilnisky (1997).

[26] Almost all of the results we discussed do not make use of the heavier homology theory. Baryshnikov's work and Koshevoy (1997) are exceptions.

[27] Kelly (1971) is one of the early contributions that investigate the continuity of the aggregated preference. See also Campbell and Kelly (1996).

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