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A Hundred Years of Numbers. An Historical Introduction to Measurement Theory 1887–1990

Part II: Suppes and the Mature Theory. Representation and Uniqueness

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In Part I we saw that the works of Helmholtz, Hölder, Campbell and Stevens contain the main ingredients for the analysis of the conditions which make (fundamental) measurement possible, but, so to speak, that what is lacking in the work of the first three is to be found in the work of the last, and *vice versa*. The first tradition focuses on the conditions that an empirical qualitative system must satisfy in order to be numerically representable, but pays no attention to the relation between possible different representations. The second tradition focuses on the study of scale types and the mathematical properties of the transformations that characterize the scales, but says nothing about the empirical facts these scales represent and the nature of such representation. Then, these two lines of research need to be appropriately integrated. In this Part II, we shall see how this integration is brought about in the foundational work of Suppes, the extensions and modifications which are generated around this work and the mature theory which results from all of this. © 1997 Elsevier Science Ltd.

6. Suppes' Foundational Work

The first author to appropriately integrate the two previous lines of research was P. Suppes, in a famous paper published in 1951 and entitled 'A Set of Independent Axioms for Extensive Quantities' where he lays the basis of the mature theory of metrization. Our statement that it is here that the two previous traditions converge is *a posteriori* and taking into account 'the matter itself'; it is not a statement about the explicit intentions of the author. Suppes does not explicitly set out to bring about such an integration, at least he says nothing suggesting such a thing. Nevertheless, whether or not Suppes was

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aware of his position in relation to previous research, the work which he starts in this article *de facto* integrates previously dispersed elements for the first time.

In this paper, Suppes claims to attempt two things: first, to find a set of conditions (weaker than Hölder's and which avoid their problems) for an empirical domain to have a morphism representation over the set of real numbers; and secondly, to study what relation there is between all these morphisms. This first work only deals with 'additive' empirical domains of the type we have seen, i.e. extensive quantities, but it allows a natural generalization to be made to other types of empirical domains.

The primitive notions are those of a domain A of objects, a binary relation Q over A, whose interpretation is 'smaller or equal in magnitude than', and a binary operation • on A of combination or concatenation. A structure $\mathbf{E} = \langle A, Q, \bullet \rangle$ is a *system of extensive quantities* (in brief, SEQ) iff (Suppes, 1951, p. 65) it satisfies seven axioms, those of positivity, solvability, Archimedianity, closure of A under • and three other ones from which (together with previous ones) it follows that Q weakly orders A as well as associativity, commutativity, connectedness and monotonicity of •. With Q, a coincidence relation C of alikeness or indifference can be defined: xCy iff$_{\text{def}}$ xQy and yQx. So defined C turns out to be an equivalence relation and the quotient set A/C is therefore a partition of A.

Suppes proves two things: if the empirical system **E** is SEQ then: (1) the quotient system **E/C** is isomorphic to an additive semigroup of positive reals, i.e. there is a one-to-one function f of A/C into Re^+ such that f is an isomorphism of **E/C** into the mathematical system $\mathbf{M} = \langle \mathrm{Re}^+, \leq, + \rangle$; and (2) every pair of additive semigroups of positive reals which are isomorphic to **E** are related through a similar transformation, i.e. if f and g are two such isomorphisms then there is $a>0$ such that for every equivalence class $[x] \in A/C$, $f([x]) = a \cdot g([x])$. The first part, which establishes the existence of a representation, will be referred to as the Representation Theorem (RT), and the second, which establishes the relation between different possible representations, i.e. to what extent or in what sense the representation is unique, as the Uniqueness Theorem (UT).[1] The reference to an isomorphism is not too strong here since the representation is proved for the quotient structure, numbers are assigned to equivalence *classes* of objects, which is the same as talking of homomorphism when numbers are assigned directly to objects. This second equivalent version, which will become prevalent, is somehow more natural since when we measure we assign numbers to *objects, not to classes*.

Now, with this step made by Suppes, the question of the admissibility of the transformations of a scale can be dealt with satisfactorily. If f is a scale-representation for an empirical system **E**, a numerical function F is an

[1]As far as I know the first place where this terminology, which was to become standard, was used was in Suppes and Zinnes (1963).

admissible transformation for f iff the result of applying F to f, i.e. the composition $F \cdot f$, is also a homomorphism of **E** into the numerical system. Mathematical properties of function F define different types of transformations, and scale types are now defined (not circularly) by reference to the type of transformation admissible for the scale, in the previous, independently specified, sense of 'admissible'. For instance, for the case above, every representation of a SME is a proportional or ratio scale, since it has been proved (UT) that the range of admissible transformations for such a representation are similar transformations. This is the required link between the two approaches we saw in Part I: what characterizes the scale type is the numerical transformation that preserves the property of 'to be a morphism (iso or homo, depending on the versions) of the empirical system into the numerical one'. So, what is relevant for the establishment of the scale type is not some purely mathematical property but certain empirical facts expressed by the conditions the empirical system satisfies and which determine the kind of mathematical transformations which preserves 'representativeness under morphism'. And if things are regarded in this way, it is quite natural to generalize this schema to other empirical systems. For it is natural to raise the question of what other systems should be like for their representations to be scales of other types. Within this framework, the work initiated by Stevens appears to be essential, since it establishes the different scale types and hence how strong the different representations can be.[2]

The above theorems require some additional comments. In the first place, RT proves only that certain conditions are sufficient for the existence of a homomorphism of **E** into **M** (although in this case they are also necessary). Second, these conditions are not categorical, since they have denumerable realizations as well as supernumerable. Thirdly, UT states that any two homomorphisms of **E** into **M** are related by a similar transformation but the converse can immediately be proved, i.e. every similar transformation of a homomorphism is also a homomorphism. Therefore, together with RT, what is proved is the following:

RUT If $\mathbf{E} = \langle A, Q, \bullet \rangle$ is SEQ and $\mathbf{M} = \langle \mathrm{Re}^+, \leq, + \rangle$ then there exists f from A to Re^+ such that for every g from A to Re^+:

g is a homomorphism of **E** into **M** iff g is a similar transformation of f.

[2]In Part I we used the terms 'numerical assignment' and 'scale' as synonymous. We can now distinguish (as is usual in literature, see for example Suppes and Zinnes (1963) where for the first time the difference is explicitly formulated) between the assignation f of A into Re and the scale itself, which is the trio $\langle \mathbf{E}, \mathbf{M}, f \rangle$. As has been made clear, knowledge of the assignment is not enough to know its uniqueness. In order to know the uniqueness, the scale must be known, i.e. it must be known the systems in relation to which the assignation is a homomorphism. Once this point has been made clear, and providing that it does not cause confusion, we shall continue to use the term 'scale' ambiguously.

As we can see, RUT has the characteristic form of uniqueness theorems: $\exists x \forall y(\varphi(y) \leftrightarrow Rxy)$, where R is an equivalence relation (if R is identity, then existence is unique). Finally, the theorems prove that *every* representation of a SEQ is a proportional scale, but not that *only* representations of an SEQ are proportional scales. So, it will be interesting to see not only if other systems have scales of other types, but also if systems other than SEQ have proportional representations.

As far as empirical applicability is concerned, Suppes' systems eliminate some of the difficulties of Hölder's but, as he himself recognizes, not all of them. The main problem is that the condition that A is closed under • implies, together with other conditions, that the domain of a SEQ is infinite (and that there are arbitrarily large entities), which, he says, flagrantly violates the obvious finitistic requisites of empirical measurement (Suppes, 1951, p. 173). There is another consequence which, even though it is not so patently undesirable as the previous one, Suppes regards as being debatable in some cases: the indifference relation C, defined from Q, is transitive. Suppes states that, perhaps because of limits to the sensitivity of the procedures to determine order, there may be cases in which two objects coincide with another but not with themselves (p. 174).

We shall discuss both these questions to a certain extent below. I want to mention now only that, not long afterwards, in his famous work *Fundamentals of Concept Formation in Empirical Science*, Hempel also points out that the domain cannot be required to be closed under the operation of concatenation.[3] Hempel takes as primitives a relation P of strict precedence, a relation C of coincidence and the combination operation •. The conditions imposed on • must be considered as only applicable to the objects 'whose combination exists and belongs to the domain' (Hempel, 1952, p. 86, f. 71). Hempel also makes two interesting points. The first is that there must be *different* coincident objects since it is necessary to dispose of standard series (consecutive combinations of coincident elements) and not every combination procedure enables an object to be concatenated with itself. The second refers to what he calls 'the condition of commensurability', according to which every object is such that it coincides with a finite concatenation of objects coincident with the chosen standard (i.e. coincides with a term of the standard series), or a finite concatenation of objects coincident with it coincides with the chosen standard. Although this condition could be considered to be empirically appropriate under certain idealizations, 'theoretical considerations strongly militate against its acceptance, for it restricts the possible values of those quantities to rational numbers, whereas it is of great importance for physical theory that irrational values be permitted as

[3]Hempel's work was published one year after Suppes', and although Suppes is mentioned in a note (fn. 71), Hempel does not follow his treatment. In particular, even though he informally reflects on the sufficiency of his conditions, at no time does he raise the questions of representation and uniqueness as Suppes does.

well' (*ibid.*, p. 68). For Hempel this proves that fundamental measurement does not give a complete definition, only a partial one, of the magnitudes. This partial interpretation has to be combined, via derived measurement, with the one supplied by the laws of the theory (in the way which became standard in the Received View in philosophy of science).[4]

Suppes' work (despite the above-mentioned problems) is the first conceptually satisfactory analysis of the conditions which make (fundamental) additive measurement possible, a kind of measurement which is sufficient, with very few exceptional cases, for physics. On the other hand, in science, especially in the human sciences, non-proportional scales are also used to measure certain properties. The question which immediately comes up is whether the type of analysis developed by Suppes is also suitable for studying the conditions which make these other forms of measurement possible. Since Suppes' general approach does not seem to depend on the specific nature of the empirical system nor on the resulting scale type, it is natural to think that the answer to this question is affirmative. During the 50s and at the beginning of the 60s a whole series of studies appeared with the aim of showing that this was indeed the case. These studies extended the formal, original nucleus of measurement theory and enlarged the domain of empirical situations accessible to it. We shall see first what is the general schema behind Suppes work and then some of its more notable extensions.

7. General Form of Suppes' Model

Suppes' general schema is simple. Let A be a set of objects to which certain numbers are to be assigned representing the 'quantity' of a particular magnitude that they have. The facts related to the magnitude are expressed by certain empirical relations $R_1, \ldots, R_n$ (some of them can be operations) between the objects. Because the objects possess the magnitude 'in a more or less degree' some of these relations will be of (some type of) order. The domain and the relations make up an empirical system $\mathbf{E} = \langle A, R_1, \ldots, R_n \rangle$ which expresses the essential nature of the property as a magnitude. Measurement assigns numbers to the objects, usually real numbers if the whole wealth of mathematics is to be applied. Empirical relations (and operations) $R_1, \ldots, R_n$ are represented by 'natural' numerical relations $S_1, \ldots, S_n$ which along with a set N of numbers (N is Re or one of its notable subsets, such as Re^+) constitute a mathematical system $\mathbf{M} = \langle N, S_1, \ldots, S_n \rangle$. The statement that numerical relations S_i represent empirical relations R_i means that $\mathbf{M}$ expresses with numbers what $\mathbf{E}$ expresses without them, i.e. that $\mathbf{E}$ is homomorphic to $\mathbf{M}$. An analysis of how measurement is possible consists, then, in studying how such a homomorphism is

[4] Later on, in Hempel (1958) (section 7, p. 62 ff.), he revises this conclusion and states that, if the underlying logical language is powerful enough (if it includes for instance the concepts of sequence and limit), it is possible to formally define metric concepts in such a way that they have irrationals as values.

possible, i.e. investigating the conditions which **E** has to satisfy for there to be a homomorphism into **M**, and establishing the corresponding representation and uniqueness theorems.

RT proves that certain conditions or axioms $Ax_1, ..., Ax_p$ are sufficient for the existence of a homomorphism and UT establishes the relation between any such two homomorphisms.[5] Taken together, what must be proved, then, is the following. Let $\mathbf{E}=<A, R_1, ..., R_n>$ be an empirical system and $\mathbf{M}$ $<N, S_1, ..., S_n>$ a particular numerical system. If **E** satisfies $Ax_1, ..., Ax_p$, then there exists a function f such that for every g, g is a homomorphism of **E** into **M** iff g is a T-transformation of f. Here 'g is a T-transformation of f' means that there is a function $F \in T$ such that $g=F{\bullet}f$ ('$\bullet$' now denotes function composition), where T is a set of functions of N into N, i.e. T is the transformation group, and 'T-transformation' names hence the transformation type (e.g. similar transformations).[6] If a particular empirical system **E** satisfies the conditions, one can proceed with the assignment-measurement, or, if it already exists, justify or establish its type. The proof of the existential part of the theorem also reveals how to carry out the assignment.

The relations and operations in **E** must be empirically feasible, although this does not affect the purely formal part of the theory. The relations and functions in **M** must be, for the above-mentioned reasons, 'natural' (Part I, Section 1). This removes a certain amount of arbitrariness by eliminating possibly 'extravagant' mathematical representations. However, it is important to point out that it does not remove all arbitrariness. The theorems assume a numerical system **M** as given. But why this particular one? There may be others which are also 'natural' and in relation to which there exists also a homomorphism. Indeed, for SEQ it is easy to show how this may be the case (in this topic, as in many others, additive measurement is a paradigmatic case of the theory). Every SEQ is homomorphic to $\mathbf{M}=<\mathrm{Re}^+, \leq, +>$ and so we have additive representations f (remember, such that $a{\bullet}bCc$ iff $f(a)+f(b)=f(c)$) which are proportional scales, unique up to similar transformations. But it is plain that they are also homomorphic to another 'natural' numerical system, $\mathbf{M}'=<(1, \infty), \leq, \cdot>$, since **M** and **M′** are isomorphic (in one direction with, e.g., the function $x \rightarrow e^x$, in the other with the inverse $x \rightarrow \ln x$). So, SEQ also has *multiplicative representations* f, i.e. such that $a{\bullet}bCc$ iff $f(a)\cdot f(b)=f(c)$. These representations are unique up to exponential transformations x^n $(n>0)$ and they are therefore logarithmic proportional scales. There is no formal reason for choosing some scales rather than others, **M** and not **M′**. It is an essential element of arbitrariness, which can

[5]Establishing the conditions and proving the theorems is a purely mathematical endeavour. The fact that such great effort is focussed on doing so is what, when studying the literature, gives the impression that research on metrization is characteristic of a purely mathematical–algebraic theory.

[6]Of course, the relation 'g is a T-transformation of f' must be an equivalence relation. This implies that each T group: (1) must have the identity function; (2) if it has a particular F it must have its inverse; and (3) it is closed under composition.

only be removed by pragmatic considerations (e.g. of simplicity; or because of historical reasons, which in our case amounts to the same thing). The fact that the reasons for the choice, which of course are important, are pragmatic and not formal suggests that, as far as the formal aspects of the theory are concerned, the *importance lies totally in the conditions that the empirical system must satisfy*.

This question will become somewhat more complicated when we look at some of the extensions of the original case. In some empirical systems, relations and operations cannot be interpreted *immediately* using a *familiar* numerical relation or function which is commonly used in the algebra of numbers. That is to say, although the qualitative empirical relations and functions have numerical interpretations, numerical relations and functions which interpret them (almost always combinations of other more basic ones) are not the ones contemplated in the well-known numerical systems which algebra usually deals with. To a certain extent, this is not a problem, since it is always possible *to define* mathematical relations, abbreviations of a combination of more basic ones, with which to form a numerical system and in relation to which the existence of a homomorphism can be proved. But in another sense, this strategy is somewhat artificial, because the numerical system thus obtained is not very 'usual' or natural. So, although it is always possible to formulate the representation theorem in the form which states the existence of a homomorphism of the empirical system into another numerical system, to do so in this way is sometimes somewhat forced.

The most general form of the theorem, of which the homomorphism version is a special case, would then be the following: Let $\mathbf{E}=<A, R_1, ..., R_n>$ be an empirical system. If **E** satisfies $Ax_1, ..., Ax_p$, then there exists f such that for every g: $\psi(g, A, N, R_1, ..., R_n, S_1, ..., S_k)$ iff g is a T-transformation of f (N is a numerical set, the S_i are 'typical' numerical relations and T is the type of admissible transformations). But now, to prevent the numerical representation being extravagant, ψ must not be very complicated, and the S_i, as well as being simple, must also be relatively basic. These requirements are really quite vague. Sure ψ must 'say' that empirical facts expressed by relations R_i are represented by relations S_i, but how to do it? One could think that this implies that ψ must, at least, contain, *for every* empirical relation R_i, one conditional (or biconditional?) of the form 'for every $x_1, ..., x_j$: if $<x_1, ..., x_j> \in R_i$ then $a(g(x_1), ..., g(x_j), S_1, ..., S_k)$' (where j is the arity of R_i and a says what happens with the numerical images under g of the objects $x_1, ..., x_j$). But even this is not always necessary. As we shall see below, the empirical qualitative facts of **E** which must be represented are sometimes quite complex facts expressed by a 'combination' of several R_i. Hence it seems there are no formal constraints, nor even very weak ones, on the form of ψ. The constraints on ψ imposed by the scientific community of MT are of a factual or pragmatic nature, in the sense that workers in MT try *de facto* to find canonic interesting representations and not extravagant or banal ones.[7] On the other hand, as in the case of additive and multiplicative representations for SEQ, we also find ourselves in this general form of RUT with the problem of alternative representations: if the theorem is true for a particular ψ and T, it may also be true for another ψ' and T' (where ψ' results from substituting one or more S_i for other S_i'), and there are no formal reasons for choosing ψ against ψ'.

8. Extensions

This general model of analysis which we have seen applied to SEQ was extended during the 1950s and at the beginning of the 1960s to other empirical systems. We do not intend to deal with all the extensions here, only with the first and most important ones. Nor shall we treat them exhaustively; in this context only their most general aspects are of interest.

8.1

The first extension to be considered is the one corresponding to the so called *difference*, or *interval*, systems.[8] For some magnitudes, such as thermometric temperature, *direct* procedures of comparison *between two objects* which display the magnitude, give rise only to ordinal scales. The main reason for this limitation is that there is no empirical procedure of concatenation for them that corresponds to (i.e. with the properties of) addition. However, if certain conditions are satisfied, representations can be found for them which are stronger than mere ordinal scales, although not as strong as proportional scales. What allows such representations is the existence, and certain properties, of a direct procedure of comparison between *pairs of objects*. One object is not compared with another but a pair of objects with a different pair. The pairs represent qualitative intervals of the magnitude for the objects in question. So, the primitive order relation is not a binary relation on A but a tetradic relation D on A (or binary on $A \times A$). The intended interpretation of $(ab)D(cd)$ is that a exceeds b in magnitude to the same or a lesser extent than c exceeds d; and, of course, this comparison is *direct*, it is not based on a previous comparison between a and b on the one hand and c and d on the other.

It can be proved (RT) that if an empirical system $<A, D>$ satisfies certain conditions, then there is a function f from A to a numerical set N (usually the set Re of reals) so that $(ab)D(cd)$ iff $f(a)-f(b) \leq f(c)-f(d)$, and (UT) that this function is unique up to linear transformations. So f is an interval scale. The key

[7]A good example of non interesting representation are merely ordinal scales. One can imagine that, if all the representations that the theory can produce were of such interest, MT would have disappeared long time ago.

[8]The first studies on this subject are Suppes and Winet (1955), Davidson and Suppes (1956), Suppes (1957, Chapter 12) and Scott and Suppes (1958). Cf. also Debreu (1960) and Luce and Suppes (1965).

of the proof is that for the pairs or intervals it is possible naturally to *define* a concatenation operation • that, together with D and $A \times A$, forms a SEQ.[9] From the proportional scale for pairs of objects a scale of intervals for the objects is easily derived.

This characterization is too general, and somehow inappropriate because of its uniformity since difference systems are subdivided in turn according to whether or not they are finite, absolute, positive or equally spaced. Although we cannot present the details here, the general idea is the following. In these systems the pairs (ab) express the difference in magnitude between objects a and b, but this qualitative difference can be expressed numerically in several different ways depending on what the properties of the empirical system are. Each pair (ab) is represented by a number $G(f(a), f(b))$, where f asigns mathematical entities to the objects and G is a *difference measure* mathematical function. The representation establishes then that there are f and G such that $(ab)D(cd)$ iff $G(f(a), f(b)) \leq G(f(c), f(d))$. Different interval systems may require different measure difference functions G. In the simplest case G is the subtraction $x-y$, but in other cases it may be the absolute value of the subtraction $|x-y|$ or even something more complicated, such as $\sqrt{(x^2-y^2)}$.

Difference systems illustrate the question with which we concluded the previous section. In interval metrization, as we have just seen, it is not exactly proved that a certain empirical system which satisfies certain conditions is homomorphic to another given numerical system. The representation is proved by proving a certain fact ψ between the components of the empirical system and mathematical relations and functions. This is not a specially drastic case because the simplicity and 'naturalness' of ψ here allows RUT to be reconverted immediately to the homomorphism version. If we define a tetradic numerical relation S on $\text{Re} \times \text{Re}$ such that $(xy)S(zt)$ iff $x-y=z-t$, then RUT proves the existence of a homomorphism between an empirical system $\mathbf{E}=\langle A, D\rangle$ and the numerical system $\mathbf{M}=\langle \text{Re}, S\rangle$. But in other cases the reconversion may not be so natural.

8.2

The second extension has to do with one of the limitations which, as we saw, SEQ had for Suppes, namely, that the induced coincidence or indifference relation is always transitive. In some situations, the indifference relation is not perfectly transitive but this does not, however, prevent its having a numerical representation. In these cases, it is inappropriate to require the primitive order relation to be a (non-strict) weak order, that is, strongly connected and transitive. In these situations the order is, so to say, even weaker, it is what is

[9]When the intervals have coincident 'borders' • is immediate: $(ab)\bullet(bc)=(ac)$; when they have not it is somewhat more complicated but equally possible.

known as *semiorder*.[10] The intended interpretation of a relation P of semiorder is '... it is *noticeably greater*[11] than ...' and its formal conditions are the following (cf. Scott and Suppes, 1958, p. 51, a simplification of Luce, 1966, p. 181): (1) not xPx; (2) if xPy and zPw then either xPw or zPy; (3) if xPy and zPx then either wPy or zPw. Semiorders (which are always strict) are between partial and weak orders (both strict), every strict weak order is a semiorder and every semiorder is a strict partial order.[12] Now, an indifference relation I can be naturally defined from P: xIy iff$_{\text{def}}$ not xPy and not yPx. Although relation I is not one of equivalence because it is not transitive, it is also possible to define from P a weak order relation R, and an equivalence relation $\mathbf{E}$, which are useful for proving the representation.

The representation of semiorders is quite special, for what is proved is (RT) that if an empirical system $\mathbf{E}=\langle A, P\rangle$ is a semiorder,[13] then there exists a function f from A to Re such that aPb iff for some $\partial>0$ $f(a)>f(b)+\partial$, that is, $\mathbf{E}$ is homomorphic to *some* numerical system $\mathbf{M}=\langle \text{Re}, >_\partial\rangle$ ($\partial>0$), where $x>_\partial y$ iff$_{\text{def}}$ $x>y+\partial$. The uniqueness of the representation here is not clear, because neither is the type of transformation which gives rise to other homomorphisms into the *same* numerical system (i.e. for the *same* ∂).

Semiorder systems are introduced to account for measurement in circumstances in which the order is not perfectly transitive. One must ask oneself, however, if all the cases of intransitivity of alikeness or indifference relations are of the same type and deserve the same treatment. Some may be due to 'the property itself' (e.g. comparison of subjective utility by means of preferences judgements) while others are only due to the accuracy limits of the comparison instruments (e.g. comparison of mass using balances). If there were sound reasons to distinguish the two cases (and I think there are), the semiorder device may be appropriate for the first case but not for the second, the analysis of which would correspond rather to the study of idealizations and the problem of error.

8.3

Another type of system is the one accounting for measurement of probability.[14] In the simplest case, unconditioned probability, the conditions refer to an order

[10]The first place in which the notion of a semiorder is introduced is Luce (1966) subsequently simplified in Scott and Suppes (1958). Krantz (1967) applies it to extensive systems (and sophisticates the representation with two bounds). Adams (1965), studies the intransitivity of ordinal, interval and extensive comparative systems, although from a different perspective.

[11]In this section we have been using the converse sense "lesser than" to refer to orders. Although in systematic studies the same mode of reference should be used, for the present historiographic purpose I shall use the mode of reference present in the literature.

[12]A strict weak order is asymmetric and negatively transitive: if not xRy and not yRz then not xRz.

[13]Suppes and Zinnes prove this for finite systems, and they leave open the question for the infinite ones (cf. Suppes and Zinnes, 1963, pp. 31–34).

[14]Works in this field goes back to De Finetti (1937). The conditions which he discusses only characterize probability qualitatively, they are not sufficient to guarantee a quantitative

footnote continued on p. 247

R on a set F which is an algebra (or an s-algebra) of sets on the set A.[15] The objects to which numbers are assigned are the elements of F, usually interpreted as events. The conditions which the system $\mathbf{E} = <A, F, R>$ must satisfy guarantee the existence of a function f of F into (0, 1) which satisfies Kolmogorov's axioms, that is, such that $<A, F, f>$ is a (finitely or denumerably, depending on whether f is an algebra or an s-algebra) additive probability space. The representation f is in this case an absolute scale, its transformation group is the identity. Any two representations are the same, i.e. there exists only one representation.

Unlike other systems, whose name does not refer to any specific magnitude, probability systems seem devoted to accounting for the measurement conditions of a particular magnitude, probability.[16] This does not pose any problem since, after all, it only means that this type of system has been established with the aim of analyzing a particular magnitude. But it is important to leave open the theoretical possibility that these systems, the same formal conditions, be applicable to other magnitudes. Which magnitude is to be measured cannot be determined by the set of conditions which define the system but by the specific empirical procedures by means of which order is established in the basic domain. A different thing is that, as a matter of fact, these conditions are satisfied only by one empirical procedure.

Probability systems exemplify, better than difference systems, the issue we referred to concerning the form of RUT. In this case it would be extremely forced to present this theorem as if it stated the existence of a homomorphism between an empirical system and a given numerical system. As we shall see, the following extensions will again confirm this point.

8.4

The next modification of the original model corresponds to what is known as *conjoint measurement*.[17] This extension aims to account for situations in

footnote continued from p. 246
measurement. Conditions which are sufficient to do so, when the base set is finite, are offered in Kraft et al. (1959) and simplified in Scott (1964). Other conditions, inspired in Savage (1954) are offered in Luce (1967) for unconditioned probability and Luce (1968) for the conditioned case. Other studies of interest are Luce and Suppes (1965) and Suppes (1969).

[15] F is an algebra of sets on A iff F is a collection of subsets of A closed under complement and union; if it is closed under countable unions it is an s-algebra.

[16] Although the nature of probability has been the subject of innumerable discussions in philosophy, here we shall not make any comment on its various interpretations, either in its objective or subjective sense. Most studies on the measurement of probability refer mainly to subjective or psychological probability but the formal conditions of these systems do not depend on it. If it is a question of subjective or objective probability it will depend on the specific nature of the procedures to establish the order R on F (judgements of the subjects, observed frequencies, etc.).

[17] It goes back to several analyses of utility carried out during the first half of the century in economic theory. Some important formal results, though from a different viewpoint than the one which concerns us here, are to be found in Adams and Fagot (1959) and Debreu (1960). The first study in the framework of measurement theory is Luce and Tukey (1964), subsequently modified in some respects in Krantz (1964), Luce (1966) and Tversky (1967).

which two attributes are simultaneously measured. In these cases, empirical comparison procedures give rise to an order between pairs of objects, each of which is regarded as displaying one of the two attributes. So the order between pairs of objects is not derived from two already known orders between each component of the pair, and a number is not assigned to each pair by combining *already available* assignments for the components of the pair. The assignments for the pair and for each of the components are obtained at *the same time*, that is the compound and each component are measured simultaneously. So in principle it is not a case of derived measurement.

Conjoint metrization analyzes the conditions which make such measurement possible. In this case the empirical systems are made up of two sets A_1 and A_2 and an order relation R between pairs of elements of both, that is, R is an order on $A_1 x A_2$. The intended interpretation of $<ap>R<bq>$ is that the 'conjunction' of the attributes in a and p exceeds or is the same as the conjunction in b and q. The conditions for the representation of the system $\mathbf{E}=<A_1, A_2, R>$ are not only those which make possible the existence of a function f of $A_1 \times A_2$ into a specific numerical set N such that $<ap>R<bq>$ iff $f(<ap>) \geq f(<bq>)$. If this were the case, it would not really be different from some of the previous ones. What is characteristic about this case is that the representation is made 'through, but simultaneously with' assignations on the A_is. The conditions have to be such that if $\mathbf{E}$ satisfies them, then there exist f_1 of A_1 into N, f_2 of A_2 into N and F of $N \times N$ into N such that $f(<ap>)=F(f_1(a), f_2(p))$, that is, $<ap>R<bq>$ iff $F(f_1(a), f_2(p)) \geq F(f_1(b), f_2(q))$. The main requirement that is usually imposed is that the attributes be essentially *independent* from each other. Independence means here that if two pairs with a common component are related under R in a certain way, they shall also be related in the same way if any other element is the common one: if $<ap>R<bp>$ for some p of A_2, then $<ap>R<bp>$ for every p of A_2, and the same with A_1. If independence is satisfied, relations R_1 on A_1 and R_2 on A_2 can be defined in such a way that they are also orders. With these orders at hand, and if other conditions are satisfied, we can establish the existence of functions f_1, f_2 and F for the representation. Among these additional conditions, a specially important one is that of *relative solvability*. The core idea this property expresses is that elements of one component have an 'equivalent' or 'projection' in the other component and also that pairs themselves have an equivalent or projection in each component. We do not give the formal expression here for, although the idea is the same for all types of conjoint systems, its formal version depends on specific features of each type.

In the way we have presented the desired representation, the problem of its possibility conditions seems almost trivial since it demands nothing of the function F.[18] Actually, desired representations are obtained for particular cases of F. The first to be studied was the one for which F is addition. The systems for which this representation is possible are called 'additive conjoint structures'. If a system $\mathbf{E}=<A_1, A_2, R>$ satisfies the conditions which define *additive conjoint structures*, then (RT) there exist f_1 of A_1 into Re and f_2 of A_2 into Re such that $<ap>R<bq>$ iff $f_1(a)+f_2(p) \geq f_1(b)+f_2(q)$, and (UT) the same biconditional is true for any linear transformations of f_1 and f_2 with the same coefficient, that is up to, respectively, transformations $ax+b_1$ and $ax+b_2$ ($a>0$); f_1 and f_2 are then interval scales related in a specific way.[19]

Additive structures are only one type of conjoint structures. Every type of conjoint representation is characterized by a specific $F(x+y, x-y, x{\cdot}y, (x{\cdot}y^2)/2, ...)$ and the different groups of conditions which make the different types of representation possible define different types of conjoint structures. This must be viewed with caution, since it allows some cases of independent representations to be fictitiously presented as cases of conjoint representation. Let us suppose that we have two magnitudes m_1 and m_2 with independent representations f_1 and f_2, e.g. mass m and velocity v. So we can define a new function $f=F(f_1, f_2)$ for a particular F, e.g. moment$=m{\cdot}v$, kinetic energy$=(m{\cdot}v^2/2)$. Formally, it seems that this situation can be reconstructed as a case of conjoint measurement whose conditions we must find,[20] for a system $\mathbf{E}=<A_1, A_2, R>$ suitable for the representation can always be construed. But the procedure is somewhat fictitious unless the relation R can be *previously* determined without any help of the orders which make f_1 and f_2 possible. If this is not possible, the type of situation described corresponds more to a case of derived measurement. It may be interesting to study the extent of its alikeness with fundamental conjoint measurement, but it is not a case of conjointness.

To conclude conjoint measurement, a final comment about its significance. When we introduced conjoint systems $\mathbf{E}=<A_1, A_2, R>$, we did not say, contrary to what should be expected, that sets A_1, A_2 represent *different* attributes the objects have. Although this is the most natural interpretation, we did not say so because it is not clear that conjoint measurement is always a case of measurement of different attributes for the same objects. It seems that sometimes what we have is one and the same attribute displayed by objects of a different type. For example, if A_1 are amounts of money and A_2 consumer

[18]Nevertheless, even if nothing is required of **F**, the question is not a trivial one. The requirement that there be only one function for each component is restrictive, and there are systems that do not comply with it (cf., in this respect, *Foundations* 1, p. 248). Tversky (1967) studies some general properties for the cases in which F is simple polynomial.

[19]For similar reasons to the ones we saw in the SEQ, it is obvious that additive conjoint structures also have 'multiplicative' representations (i.e. there are $f_1'...f_2'$ ... such that ... iff $f_1'(a){\cdot}f_2'(p) \geq f_1'(b){\cdot}f_2(q)$) unique up to transformations a_1x^n and a_2x^n ($a_i>0$, $n>0$) (f_1' and f'_2 are logarithmic interval scales).

[20]In *Foundations* 1, pp. 246 and 267 the case is presented as an example of conjoint measurement. Later on, however, the authors make some considerations that are similar to ours (p. 277).

goods (or, in general, two different types of consumer goods) and R is a preference relation, f_1 and f_2 measure the utility of the objects of each type and $f=F(f_1, f_2)$ (for a particular F of Re^2 into Re) measures the utility of pairs of objects. It is not clear in this case that the different utilities are different attributes. Perhaps the most natural interpretation would be to regard F as expressing a law that establishes the relation between utilities (in both cases the same attribute) of objects of different kinds, a law which relates the utility of components with the utility of the compound.

This suggests a further caution against fictitious cases of conjoint measurement. If we have a domain of concatenable objects, the concatenation of two objects could be interpreted as a pair of objects to be measured conjointly. For example, it can be proved that if $<A, Q, \bullet>$ is a SEQ then the system $<A, A, R>$, with R defined so that $<xy>R<zw>$ iff $z \bullet w Q x \bullet y$ is an additive conjoint structure.[21] It seems quite clear, however, that to describe such a case as one of conjoint measurement is misleading.

8.5

The next extension of the model concerns the mathematical system into which the representation is carried out. So far, the mathematical entities assigned to the (simple or complex) empirical objects were always numbers. However, we have already seen that Helmholtz had called attention to certain cases in which the same physical combination operation gives rise to the simultaneous conjunction of various magnitudes, and he mentioned 'vectorial magnitudes' as typical examples, taking each of the components of the vector as one magnitude. These cases belong to what in the present context is called 'multidimensional representation'.[22]

In multidimensional representation the mathematical entities assigned to the objects are n-dimensional vectors on Re (when $n=1$ we obtain the usual numerical representation). The mathematical structures **M** are in this case (vectorial) n-dimensional numerical systems with a domain V of n-dimensional vectors and some relations and functions on V.[23] The formal scheme of the representation does not vary: if **E** is of a particular type, then it is homomorphic to a specific **M** system; in general, if **E** satisfies certain conditions, there is f of A into a set V of vectors such that a certain empirical state of affairs between objects of A occurs iff a certain vectorial state of affairs occurs between their f-images. Despite the formal resemblance to genuine metrization, not every case

[21]This possibility has already been contemplated in measurement literature, informally in Luce and Tukey (1964, sec. VII) and, formally, in Narens (1985, p. 174); see also *Foundations* 3, p. 81.

[22]The first place in which it is studied from this perspective, although on a very superficial level, is in Suppes and Zinnes (1963, pp. 47–48). The analysis emerges from vectorially reinterpreting the numerical representation of some of Coombs' preference systems (cf. Coombs, 1950, 1960; Bennet and Hays, 1960).

[23]Of course functions must be internal (operations like scalar product or the norm cannot represent empirical operations between those objects to which vectors are assigned).

of multidimensional or vectorial representation can be regarded as a real case of representation of magnitudes (properties capable of instantiation degree) and, hence, as relevant to our Measurement Theory. When we discuss, below, the vectorial part of the second volume of *Foundations*, we shall see the reasons that, in my opinion, militate against including the vectorial representations which analytic geometries make of synthetic (i.e. qualitative) geometries as genuine cases of metrization.

8.6

The last important family of modifications we shall consider has to do with the empirical operation of combination •. We saw that in SEQ one requirement was that A is closed under • and that this requirement (together with other reasonable ones) has the unpleasant consequence that there are no finite SEQ. For extensive systems to be finite their conditions must be weakened, in particular closure under •. This can be done basically in three ways. First,[24] we can substitute in E: (i) the operation • by a set F of subsets of A (closed under union and complement); and (ii) the relation Q on A by another relation R on F. The idea is to take the elements of F as the objects-arguments for assignment (F always has the 'concatenation' $\{x, y\}$ of two objects $\{x\}$, $\{y\}$ belonging to it). If such a system satisfies certain conditions (which now does not imply that there are necessarily infinite objects) the desired additive representation, i.e. $f(\{x, y\})=f(\{x\})+f(\{y\})$, can be found (in general, for A, B in F with $A \cap B=\emptyset$, $f(A \cup B)=f(A)+f(B)$). The second possibility[25] is to substitute • by a ternary relation, which is a function but not necessarily defined for every pair of members of A, and give appropriate conditions for making the representation possible. The third way[26] , similar to the previous one, consists of keeping • in the system but relativizing its conditions to a subset B of $A \times A$ which contains the pairs of elements whose concatenation exists.

These modifications again illustrate the comment we made above about the way in which RUT is formulated. It is clear that to present the possibility of representation for these systems as being homomorphic to a numerical system is, because of the artificial nature of the numerical systems required, absolutely unnatural. For cases such as these it is more natural to present RUT in its general form, namely, if the empirical system satisfies certain conditions then there exists an assignment such that certain 'natural' statement (about the empirical relations, the assignment and mathematical relations) is true.

[24]As far as I know, the first place where extensive systems are presented in this way is Adams (1965) (he discusses the reasons on pp. 207–208). Other places where this idea is taken up are Krantz (1967), Suppes (1969, pp. 4–8) and Suppes (1972). As the reader will appreciate, the empirical systems here are of the same type that we saw previously for probability systems. Adams and Krantz do not link both types of system, but Suppes does.

[25]Suggested in Suppes and Zinnes (1963, p. 45) and developed in Luce and Marley (1969).

[26]This is the way that will be followed in *Foundations* (cf. Vol. 1, sec 3.4).

The closure of • is inadequate to account for many empirical situations, and so the above modifications are necessary. But the fact that most cases of additive conjunction require such a weakening does not imply that the closure condition is always inadmissible. On the contrary, there are situations in which not only it is not inadequate but it is necessary. This is the case of 'periodic' magnitudes such as angles. The conditions which must be satisfied for the appropriate representation to be possible in these cases define a new type of system, the extensive *closed periodic structures* (cf. Luce, 1971). The representation in this case is periodic in a cycle, and 'additive' if we take the addition in the modular sense: $f(a{\bullet}b)=f(a)+_a f(b)$ (with $x+_a y=z$ iff$_{\text{def}}$ there is an integer n such that $x+y=(n{\cdot}a)+z$, that is, z is the remainder of $x+y$ divided by a). Another type of extensive system, the *essential maximum structures*,[27] accounts for cases in which the (non-closed) concatenation operation has a limit or maximal element, i.e. there is an object such that it is never exceeded by the concatenation of others (e.g. for relativistic velocity, the speed of light).

In the last two cases, although the representation still has a certain spirit of additivity, it is not additive *stricto sensu* (i.e. $f(a{\bullet}b)=f(a)+f(b)$). They are only some of the systems with essentially non-additive representations (i.e. such that $f(a{\bullet}b)=F(f(a), f(b))$, where F is not the addition).[28] Different conditions for the physical combination • give rise to other non-additive systems. Of particular interest are cases such as the combination of temperatures or densities, in which • is not positive but *idempotent* ($a{\bullet}a$ coincides with a) and *internal* (if a and b do not coincide, their combination lies between them).

Up to this point • has always been interpreted as a physical combination (although in some cases it does resembles numerical addition, or even does not satisfy positivity). Nevertheless, there may be operations on the domain of objects which are empirically meaningful and which make interesting representations possible but which cannot be interpreted as a combination of objects in any reasonable sense of the term. *Bisection (or bisymmetric) systems* are a paradigmatic case of this situation.[29] In such systems the intended interpretation of $a{\bullet}b$ is 'the mid-point between a and b' (e.g. the subject is asked to choose a stimulus which is equidistant from two given ones). The representation in these cases becomes quite complex and, again, it is completely unnatural to present it as the possibility of a homomorphism into a given numerical system.

These are the main modifications of the original Suppesian model. The review has been very schematic, each modification gives rise to a whole family

[27]Cf. Luce and Marley (1969). In these systems the Archimedean axiom must be relativized to non-maximal elements.

[28]'Essentially' because, as we saw, there are systems (e.g. SEQ) with both additive and non-additive representations. Another typical case of essentially non-additive representation is the combination of resistances in parallel, for which F is $(x{\cdot}y)/(x+y)$.

[29]The bisection method has a long tradition in psychophysics. Pfanzagl (1959) introduces, by reformulating some of the Aczéls' (1948), bisymetrical operations (of which the psychophysical procedure of bisection is one) and the corresponding structures.

of increasingly complex cases and there are also mixed and cross-linked cases. We do not intend to go into these additional complications now. What we have already seen is enough for our present introductory purposes before tackling the mature theory.

9. The Mature Theory

Research on metrization crystalizes, from the end of the 1960s, in a series of works that systematize, organize and, in some cases, extend the previous results. The most important is undoubtedly the *opus magnum*, *Foundations of Measurement*: (1971, 1989 and 1990 respectively each of its three volumes—Krantz et al. (1971), Suppes et al. (1989) and Luce et al. (1990)). Others which also deal with the conditions that make fundamental measurement possible are Ellis (1966), Pfanzagl (1968), Roberts (1979), Berka (1983), Kyburg (1984) and Narens (1985).[30] Of course, we do not intend to summarize, even briefly, these studies. Because the relevance of these works for our story is variable, we shall only point out the most important aspects for our present concerns.

Basic Concepts of Measurement by Ellis (1966), is one of the few studies on measurement with a more philosophical than mathematical approach. Ellis' general aim in this study is to attack a certain metaphysical realist view of magnitudes arguing in favour of the *essentially relational* character of metric concepts.[31] This is not the place to discuss this issue, but some comments which he makes (which are advanced in Ellis (1960, 1961) are of interest for the analysis of fundamental metrization. Ellis defends (p. 32) the view that the identity criteria for magnitudes come from the order relation, extensionally considered. Several logically or intensionally independent ordering procedures may correspond to the same magnitude, generate the same extensional order; hence metric concepts are 'cluster concepts' which cannot be defined by reference to particular ordering procedures. If this is so, then measurement is more arbitrary than is commonly admitted. The arbitrariness of, for example, extensive magnitudes is not limited only to the choice of the unit or standard, since they have scales which are not related by a similar transformation. The reason is that, as we are going to see, different modes of additive combination can exist for the same magnitude.

Campbell pointed out that both the combination of wires in series and in parallel are additive, but that they are so with respect to different orders (in this case converse orders), and therefore they are additive combinations of different (although related) magnitudes. Ellis describes a case in which two different

[30]As in *Foundations*, the works by Roberts and Narens deal exclusively with fundamental metrization. In the others, this topic is treated as part of a broader analysis of measurement.

[31]Recently (Ellis 1987) he has abandoned this position and has become a defender of a certain type of metaphysical realism about 'quantitative properties', similar to that of Swoyer (1987) (cf. for this topic, also Forge (1987), Armstrong (1987, 1988)).

modes of combination are both additive with the same order and so, according to him, for the same magnitude (Ellis 1966, p. 79; cf. also Ellis 1960, pp. 44–46). The example he gives is about length. Let • be the usual linear combination of rods. Let •′ be their orthogonal combination.[32] Let R be the usual order: aRb iff when putting a and b on a straight line with their origins coincident, the end of b coincides with or is after the end of a. A domain A of objects with an order R constitutes an extensive system both with • and with •′, so there are for these systems, respectively, representations f and f' both additive in the very same strict sense, i.e. the assignment to the compound is the sum of the assignments to the components. But now f and f' *are not related by a similar transformation*, f' is not proportional to f but to f^2.[33] According to Ellis, because the magnitude involved here is the same as the order, we have as a result two non-proportional scales for the very same magnitude, length. The mode of combination is an element of arbitrariness which cannot be eliminated.[34] The choice of one of the modes, and therefore of one of the types of scale, can only be based on reasons of simplicity; the whole of physics could be rewritten with f' instead of f, the *only* difference would be in complexity (p. 82; but he adds that, nevertheless, even some parts of physics would be simplified by taking f').

Ellis also poses another question, which he calls 'the second problem of fundamental measurement' (cf. 1960 and 1966, pp. 86–88), and which we shall only mention because it has to do more with measurement than with metrization. Once the mode of combination has been chosen, the standard must be chosen before proceeding with the assignment. Generally speaking, the choice of the standard is regarded as being unproblematic, but this is not at all evident. Let us suppose a world in which the objects which display a particular magnitude are divided into two groups in such a way that the elements of each set, with respect to the magnitude, behave stably with each other but unstably with the elements of the other set. In such a world, the form of physical laws would be affected by the choice of the group to which the standard belongs, and, once again, the only reason to choose one or another, would be simplicity.[35]

[32]The 'resulting rod' in this case is not straight but this does not matter; if it is required to be straight, the resultant can be considered to be the straight line which joins the origin of one with the end of the other, that is, the diagonal of the resultant in the previous sense.

[33]We have for $<A, R, \bullet>$ a function f such that $f(a \bullet b)=f(a)+f(b)$ and for $<A, R, \bullet'>$ a function f' such that $f'(a \bullet' b)=f'(a)+f'(b)$. It is easy to see that for every $a \in A$ $f'(a)=f^2(a)$. (This does not mean, of course, that $f'(a \bullet' b)=f^2(a \bullet b)$, but $f'(a \bullet' b)=f^2(a \bullet' b)$; it is obvious that $f'(a \bullet' b)=f^2(a \bullet b)-2f(a)f(b)$.)

[34]Of course, to describe this situation as the existence of two non-proportional scales for the *same magnitude* it must not be considered that only one of the modes of combination is essentially linked to the magnitude. If every magnitude had only one mode of combination essentially associated to it there would not be non-proportional different representations of the same magnitude.

[35]Conceptually, this question is the same one that Hempel refers to (Hempel, 1952, pp. 73–74) when he mentions (attributing the example to Schlick) the possibility of taking the Dalai Lama's pulse as the standard for the measurement of duration. This problem was also considered by Carnap (cf. Carnap, 1966, Chapter 8.)

Ellis (1966) does not make any substantive contribution to the formal conditions which make measurement possible, nor does he deal exhaustively with the various systems which these conditions define. The first to gather together and present most of the results that we saw in the previous section was Pfanzagl. Pfanzagl (1968) is the first work which systematically studies the different algebraic properties of empirical systems and possible types of representation.

In the presentation of the systems, instead of a single 'less than or equal to' relation, Pfanzagl uses two relations, one $<$ of strict order and one $\sim$ of coincidence, which is an equivalence relation (therefore he does not deal with semiorders). He also studies and presents in a general way a whole series of operations (only some of which can be interpreted as combination) and the systems which they give rise to. His presentation of the systems is sometimes peculiar[36] and RUT is always offered in the homomorphism version, so the mathematical systems which are taken for the theorem are often really very unnatural.[37] On the other hand, for most of his systems he requires excessively strong structural conditions (which enable him, for example, to do often without the Archimedean axiom).

Apart from it being the first *summa*, this work is important for some specific results. It studies interval scales based on operations (Chapter 6) and the relation between systems with operations and the usual difference systems (Section 9.2). It gives (Chapter 7) psychophysical applications of some of the operations studied, among which are those that 'divide' a stimulus in half (*middling*) and interpolate a stimulus between two others (*bisection*), and the corresponding generalizations for division into n parts and n-interpolation. It generalizes systems of conjoint measurement[38] for cases in which there are more than two attributes to be measured simultaneously (*k-dimensional conjoint measurement system*, p. 149) and proves that the case with three or more compounds is essentially different from the usual one with two compounds (p. 140). Finally, his analysis of the empirical status of some of the axioms should be given special mention (Sections 6.6 and 9.5); this analysis had been initiated by Adams, Fagot and Robinson in a 1965 study which is the origin of Adams *et al.* (1970).

Pfanzagl's work is quite unsatisfactory in completeness of treatment and systematism of presentation. The same cannot be said of *Foundations of*

[36]E.g. he defines difference systems with two order and two equivalence relations, one between pairs of objects and the other between objects (p. 143), or he introduces into the systems a relation L of 'limit of a sequence' (p. 78).

[37]E.g. theorem 6.1.1.; the version of RUT with a homomorphism must be understood as being a mere abbreviation, defining a numerical system, of a more complex formulation, not as the 'alikeness' of an empirical system with another mathematical natural system.

[38]In Chapter 12 Pfanzagl analyzes simultaneous measurement of utility and subjective probability in detail, but its relation with the general analysis of conjoint measurement (Chapter 9) is not clear.

Measurement, the first volume of which was published in 1971. The second (and almost mythical) volume was finally to appear twenty years later divided into two more parts. This overwhelming work (1500 pages and nearly 400 definitions and main theorems) is undoubtedly intended to become the essential work of reference on metrization. Nevertheless (and apart from the fact that its enormous amount of information makes it sometimes difficult to read in a unitary way), it is not always clear which parts are directly connected with measuring conditions of an empirical system or with other different but related topics.[39]

The first volume basically deals with extensive, difference, probability and conjoint systems. The basic order relation is $\succcurlyeq$, whose intended interpretation is 'greater than or equal to' (semiorders, which as we saw cannot be analyzed by this type of order, are studied in a later volume and in the context of error). In extensive systems, $\succcurlyeq$ is on a domain A and a concatenation operation • is also added. Several possibilities are studied, depending on whether the operation is closed or not, whether it has maximums or not and whether the order is connected or not. In difference systems, $\succcurlyeq$ is on the Cartesian product $A \times A$ of the basic domain A, and they are distinguished depending on whether the intervals are positive, absolute or equally spaced. About probability systems, the authors study those which correspond to unconditioned and conditioned probability and some modifications are introduced for specific cases. Finally, in conjoint measurement, first they study conjoint systems with two components, mainly additive systems but also those with essentially non-additive representations, and afterwards the results are generalized to systems with n components, with both additive and, generally speaking, polynomial representations. The relation $\succcurlyeq$ is in this case on the product $A_1 x \ldots x A_n$ of the basic domains A_i. RUT does not have always the form of the existence of a homomorphism; when the numerical systems would be very unnatural, what is proved is the existence of a numerical assignment that complies with certain conditions.

We cannot here discuss this scheme, or even mention the main contributions of the various parts of the book. I shall confine myself to mentioning just one especially interesting point concerning the general form of the Archimedean condition. As we have seen, the Archimedean axiom requires no element to be 'infinitely greater' than another. In extensive systems, for example, this means that if b is greater than a, concatenating a with itself (or with some other one similar to a) a finite number of times, we can 'reach' or surpass b. In other words (and defining na recursively as follows: $1a=a$, $na=(n-1)a$•a): if b is greater than a, the set of integers n such that b is greater than na, is finite.

[39]Cf e.g. Chapters 12 and 13 on multidimensional representation or Chapter 16 on threshold representations; for a review and general assessment of Volumes 2 and 3 of *Foundations* from the perspective of a theory of metrization, cf. Diez (1993).

Expressed in yet another way: should the condition not be complied with, then there is a sequence a, $2a$, $3a$, ... which is strictly bounded, by b, and infinite. A similar condition must be a requirement in other systems.[40] In Vol. 1 of *Foundations*, a general form of the Archimedean axiom is given which has the same form for all systems, although it is relativized to a certain element which changes from system to system. The general form, which should be clear if we heed the third of the characterizations we have given for the extensive case, is: Every strictly bounded standard sequence, i.e. progressive and equally spaced, is finite. Now it is only necessary to specify what a standard sequence is in each case. We already know it for extensive systems. For difference systems, for example, it is a sequence a_1, a_2, ... such that each term forms with the previous one an interval which is equivalent to the one it forms with the following term (remember that in this case the order, and consequently the derived equivalence relation, is between pairs or intervals). In this manner, *Foundations* 1 establishes the idea of Archimedean property in a general way for different measurement systems.

Of the works which appear in the almost twenty years between the first volume of *Foundations* and the two last volumes, the most important is Narens (1985). Before considering this work at some length, let us mention a few other writings of this period. Roberts (1979) is basically an excellent up-to-date exposition of previous results which pays special attention to applications in psychology and the social sciences. Roberts always uses a strict (i.e. ireflexive) order R as a primitive relation for the empirical systems, and with R he defines an indifference relation I (aIb iff not aRb and not bRa) which, generally speaking, although not always (remember semiorders), is one of equivalence.[41] Berka (1983) is an analysis of measurement in general which does not contain new elements of interest for our present concerns. Berka mainly makes a critical review of measurement literature, but sometimes he also defends his own philosophical thesis, explicitly grounded (cf. e.g. pp. 53 and 217) on a Marxian perspective, which we are not going to deal with here. Kyburg (1984) gets away from the representational theory of measurement, at least from the common way in which the theory has developed. Kyburg gives a central role in his analysis to the topic of error, which forces, according to him, the traditional approach to be refocussed since it is incapable of dealing adequately with such a phenomenon. Exactly how he develops his theory using the construction of two 'languages' does not interest us here. The essential thing is that the theory is construed on

[40] About the possibility of representation without Archimedianity, see below the comments on Narens (1985).

[41] This procedure is almost always equivalent to the one followed in *Foundations*. 'Almost' because in *Foundations* there are some systems, cf. Section 3.12, in which the 'greater than or equal to' relation is not necessarily connected, and in Roberts this relation ('R or I') is of course connected, given the definition of I.

two levels, one which includes judgements on observed relational states of affairs and the other which expresses certain idealizations of these states of affairs. The conditions with which traditional representational theory characterizes the various empirical systems are on this second level,[42] so strictly speaking—according to Kyburg—such systems would not after all be empirical.

Narens (1985) book *Abstract Measurement Theory* is undoubtedly the other basic reference in measurement theory and the best example of how the theory sometimes acquires a purely mathematical appearance. Besides impeccably presenting most of the results which we already know, this work contains more material, much of which appeared at the end of the 1970s and beginnings of the 1980s as a result of the collaboration between Narens himself, Luce and Cohen. We shall mention here only the most important results.

The first one (cf. also Luce and Narens, 1985) is the generalization of extensive systems of *Foundations* to cases in which concatenation is not commutative and/or associative. The result is a very general type of structure, the *positive concatenation* structures. These may become even weaker if positivity is not required, which seems desirable in some cases, specifically, when the combination operation is *intensive*.

The main novelty included in the book has to do with a new way of characterizing the various systems and their relation to scale types (cf. also Cohen and Narens (1979), Narens (1981a,b), Luce and Cohen (1983) and Luce and Narens (1985)). The basic idea is to define the type of a system not directly by giving a list of axioms about the constituents of the systems, but by means of certain properties that the group of automorphisms of the system satisfies.[43] These properties basically refer to the number of points (parameters or *degrees of freedom*) necessary for the preservation of certain facts (which must be specified) in the group. The properties are *uniqueness* and *homogeneity*. The group of automorphisms of a system $\mathbf{X}=\langle X, \succcurlyeq, R_2, R_3 \ldots \rangle$, where $\succcurlyeq$ is a total order, satisfies n-uniqueness iff for every automorphism f, g, if there are different $a_1, a_2, \ldots, a_n$ in X such that $f(a_i)=g(a_i)$ (for every $i=1, 2, \ldots, n$), then $f=g$; i.e. any two automorphisms which coincide on n different points coincide on the rest, are identical. The group satisfies *n-homogeneity* iff for any two series of n different points $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $a_1 > a_2 > \ldots > a_n$ and $b_1 > b_2 > \ldots > b_n$, there is an automorphism f in the group such that $f(a_i)=f(b_i)$ (for every $i=1, 2, \ldots, n$), i.e. for any n different points and for any other n different points there is an automorphism in the group which 'moves' the first ones to the second, with

[42]This remark is related with, but has greater scope than, the discussion in representational theory about the descriptive or normative character of the axioms for some relations, such as the preference between goods or events, typical of the social sciences (cf. e.g. Roberts, 1979, pp. 3–4).

[43]An automorphism of a system **E** is an isomorphism of **E** *onto* **E**. If it is a homomorphism *into* itself then it is an endomorphism; all automorphisms are, hence, endomorphisms.

the only condition that the second ones preserve the order of their originals. Different types of system are then defined by reference to these properties: **X** is a ... structure iff its automorphism group is ...-homogeneous and ...-unique; for example, **X** is a *scalar structure* iff its group of automorphisms is 1-homogeneous and 1-unique (Narens, 1985, def. 2.4.6, p. 48).

The representation and uniqueness theorem now takes a special form, establishing the relation between the degrees of freedom of a system and its representation. Firstly, scale types are abstractly defined (as before, scale types are linked to transformation types): let **X** be a relational system, F is a scale for **X** of type a iff there is a concrete numerical system **M** (no matter how unnatural it may be) such that: (i) F is the set of all homomorphisms of **X** into **M**; (ii) F is not empty; (iii) if f is in F then every T-transformation of f is in F (i.e. F is closed under T-transformations); and (iv) if f and g are in F then one is a T-transformation of the other (i.e. all the elements of F are equivalent according to the T-transformation). For example, for **M**=<Re$^+$, $\geq$, ...> and T being similar transformations we obtain the definition of the *ratio scale* for **X** (def. 2.4.1, p. 42); as can be seen, now scales are not homomorphisms but sets of them. RUT now has the following form: if **X** is an n-homogeneous and m-unique structure (for certain n, m) then it has a scale of type a (for a certain type a); sometimes, with additional conditions, we may also prove the converse. For example, **X** is a scalar structure (i.e. its automorfism group is 1-homogeneous and 1-unique) and all of its endomorphisms are automorphisms iff there exists F such that F is a ratio scale for **X** and some element of F is *onto* Re$^+$ (theorem 2.4.4, p. 49).

This characterization of the systems is more general than the traditional one (which Narens also deals with) in the sense that *different* systems in the traditional sense may have the *same* properties of homogeneity and uniqueness. The reason is that these properties are more closely linked to representational possibilities than the traditional ones. These results, and the approach based on analysing the groups of automorphisms on which the results are inspired, are undoubtedly of great mathematical interest, but their function in a non-purely mathematical measurement theory is debatable. Not only because they require formal assumptions that are too strong (such as total ordering), but mainly because it is difficult to find a *direct* empirical interpretation of homogeneity and uniqueness properties *without* 'passing through' the traditional ones.

Narens does not always follow the strategy based on analyzing groups of automorphisms and most of his work fits the traditional approach. This part of his work follows, in general, the lines that are familiar to us, and I shall not insist on this here. There is, however, one part of it that is new and that, though worthy of a more detailed discussion, I shall only mention. It is the possibility of representing structures in which the Archimedean condition, or any other stronger condition, is not required (cf. before Narens (1974) also Skala (1975)). The motivation for studying this type of system is two-fold. On the one hand, this condition is not first-order axiomatizable;[44] on the other, in some systems it may be reasonable to admit the existence of infinite or infinitesimal elements, that is, infinitely bigger or smaller than others.[45] Narens studies several extensive and conjoint non-Archimedean systems. These systems are not generally representable on the reals, but they are on a ultrapower of the reals and the last part of the book deals with this kind of representation following the techniques of non-standard analysis.

Many of the results contained in Naren's work are included, extended and brought up to date in the third volume of *Foundations*. Here extensive systems are generalized by using concatenation structures (Chapter 19)[46] and the new characterization of the systems using the properties of their automorphism groups and their relation to scale types (Chapter 20) is presented. Of the two remaining chapters in this third volume, Chapter 21 is not, with respect to MT, so much theoretical as metatheoretical, for it deals with several issues concerning axiomatizability. Actually, most of it is simply standard model theory and axiomatization in formal languages, and only at the end (cf. 21.7 and 21.8) is the topic applied to MT. The last chapter is devoted to invariance (a subject which had been partially dealt with in *Foundations* I, Chapter 10) and the problem of meaningfulness. Three concepts of invariance, *referential, structural* and *transformational*, are introduced and the importance of each one and the logical relations between them are discussed. The notion of invariance is directly linked to the meaningfulness problem, that is, what legitimate or *meaningful* use can we make of representations-measurements? Once we have a representation of a certain system-magnitude, not every quantitative statement involving the magnitude is legitimate or meaningful, in the sense of depending *only* on 'the objective facts of the system', and not on our conventions. So, for example, 'the probability of this event is 0.7' is meaningful, but 'the mass of this object is 4.3' is not; and 'the mass of this object is 1.5 times the mass of that one' is meaningful whereas 'the (thermometric) temperature today is twice yesterdays' is not. The first method of tackling this question, which goes back to Stevens, made meaningfulness depend on invariance under the admissible transformations of the morphism-representation, i.e. under changes of scale. In

[44]This is a consequence of the compactness theorem (cf. Narens, 1985, p. 318 ff.). As is pointed out in *Foundations* 3 (p. 248) the importance of this fact for measurement theory must be qualified, because if, as is usual in metrization, set theory is taken as the basic language, the Archimedean axiom is on a par with the others.

[45]Kyburg (1988, p. 182) relativizes the importance of this possibility. On the other hand, Narens deals with this possibility in general. It has nothing to do with some concrete systems with 'insuperable' elements, such as systems with an essential maximum which we have seen above (see Section 3.6).

[46]This generalization and systematization is, however, still partial; for a posterior canonical reconstruction of all types of combination systems, cf. Díez (1997, Chapter 6) and Moulines and Díez (1997).

this chapter, the authors review the evolution of this criterion and other subsequent ones, and discuss its application in the two main areas studied in measurement literature, dimensional analysis and statistics.

The second volume of *Foundations* is devoted mainly to analyzing multidimensional representation (Chapters 12–15) and error (16–17). About the latter, rather than with error in the representation properly speaking, they deal with the various representational possibilities for systems in which some central condition of standard cases fails. The classical case and the one on which they focus is transitivity, which we discussed in Section 8.2 and with respect to which the authors add here a more probabilistic approach (Chapter 17) to the usual algebraic analysis of these systems. This part on the theme of 'error' contains few new elements worthy of mention. More interesting, because it reveals the extremely, and sometimes exceedingly, broad concept the authors have of a theory about the foundations of measurement, is the section on what they call *multidimensional representations*. We shall conclude with some comments on this issue.

After what we have seen up to here, one would expect multidimensional metrization to deal with the conditions that an empirical system $\mathbf{E}=<A, R_1, ..., R_m>$ must satisfy to be homomorphic to a certain vectorial system, or geometry, $\mathbf{V}=<\mathrm{Re}^n, S_1, ..., S_m>$; or, in general, for there to be a function f of $\mathbf{E}$ into Re^n for which a ('natural' and interesting) formula ψ relating f, the R_i and certain relations and operations on Re^n, is true. However, this is found in only a very few places in the four chapters which make up this section.

First, the various representative structures, i.e. the various analytical geometries, are introduced.[47] Then, several synthetic geometries, i.e. systems which are characterized using qualitative axioms about relations on the qualitative domain, are presented (e.g. projective, affine, absolute, Euclidean, hyperbolic, elliptical spaces). It is proved that they are isomorphic (in each case) to a specific analytical geometry. This is where we find the above-mentioned scheme most clearly. To what extent this topic properly belongs to a theory of *measurement* of *empirical systems*, or whether it is simply a task of reducing synthetic to analytical geometries, is debatable. It is true that it fits the general pattern of the formal part of the theory: if a certain qualitative structure satisfies certain conditions, then there is a (now multidimensional) representation which is unique to a certain extent. A different issue is whether such systems have an empirical application and, above all, whether their representation can be properly regarded as a genuine case of *measurement*, that is, a representation of *magnitudes*. In this respect, it is important to point out that the qualitative structures here represented, the synthetic geometries, lack a

[47]Indeed, Chapter 12 is basically an analysis of the axiomatic foundations of analytical geometry. At least one of these geometries is somewhat peculiar as 'representor' system, for it contains an external operation (Minkowski geometry, def. 12.9, p. 43; the operation is the norm).

comparison relation. But this seems quite strange from a measurement point of view. If we are talking of measurement in the strict sense, i.e. quantitative representation of qualitative magnitude systems, and if magnitudes are characterized as being properties with 'more or less' instantiation or instantiation degrees, it should be expected, contrary to what is the case in geometrical representations, that every qualitative system 'containing' a magnitude essentially exhibits a primitive order comparison relation. Actually, nobody thinks that in geometrical representation we are representing magnitudes in the sense here specified. What is going on is that the authors have in mind an extremely broad, and in my opinion inappropriate, idea of what a measurement theory is, for they seem to put every kind of quantitative representation of an empirical system in the very same bag (cf. on this topic the historical comments they make in *Foundations* II, pp. 1–2). Although it is certainly interesting to develop a General Representation Theory,[48] with measurement and geometric (and perhaps other) representations as special cases, I think that MT is not such a theory, for not every case of representation is a case of measurement in the strict, I mean appropriate, sense of the word, i.e. a case of representation of *magnitudes*.

One last question should be mentioned, the surprising inclusion of the structures for the measurement of proximity or distance (cf. *Proximity Measurement*, Chapter 14) in this part devoted to multidimensional metrication. These structures are of the type $<A, \succcurlyeq>$ with $\succcurlyeq$ on AxA (of the same type, therefore, as the difference systems we saw in Section 8.1). Their conditions guarantee the existence of a function ∂ of AxA into Re which preserves order and, among other things, is a metric, i.e. (i) $\partial(x, x)=0$, (ii) $\partial(x, y)\geq 0$, and (iii) $\partial(x, y)+\partial(y, z)\geq\partial(x, z)$. When A is factorial, that is $A=A_1x...xA_n$, the representation may in some cases 'decompose'. This can be done in two ways. Some groups of conditions guarantee the existence of functions f_i of A_ixA_i into Re and F of Re^n into Re such that $\partial(a, b)=\partial(<a_1, ..., a_n>,<b_1, ..., b_n>)=F(f_1(<a_1, b_1>), ..., f_n(<a_n, b_n>))$. Other groups guarantee that there are functions g_i of A_i into Re, G of Re^2 into Re and F of Re^n into Re such that $\partial(a, b)=\partial(<a_1, ..., a_n>, <b_1, ..., b_n>)=F[G(g_1(a_1), g_1(b_1))], ..., G(g_n(a_n), g_n(b_n))$ (sometimes ∂ is no longer a metric although it still preserves order). If any of these cases can be described as multidimensional it is surely the last one, since it may be considered that we assign to each element of A an n-tuple of reals, an n-dimensional vector $<x_1, ..., x_n>$, and that the 'distance' between two elements is the result of operating in a specific way with their assignations-vectors (here x_i is the value that g_i assigns to the ith-component of the element; remember A is factorial). Anyway, it is perhaps more appropriate to describe this case as a special type of interval-conjoint metrication; and indeed most of the tools used in its analysis do come from both types of metrication.

[48]See, for instance, the works by Mundy, specially Mundy (1986).

Here we close the review of the most significant contributions of *Foundations*. In this *opus magnum*, the theory whose first stone was laid by Helmholtz a century before, reaches full maturity. But, surely enough, it still has to progress in different ways in the future. As in any other theory, normal science, in the Kuhnian sense, does not stop for MT, and since the not so remote publication of the last volume of *Foundations*, there is already an enormous amount of new writings on the subject. But, despite this impressive production, a long time is sure to pass until we see another *summa* comparable to *Foundations*. Although some of its parts deviate a little from a theory about the foundations of *measurement*, in the strict sense of the word, and others should have a better metatheoretical structure, there is no doubt that this book will be for a long time the basic reference for anybody interested in the theory of measurement.

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