WHICH SCORING RULE MAXIMIZES CONDORCET EFFICIENCY?

DAVIDE P. CERVONE, WILLIAM V. GEHRLEIN, AND WILLIAM S. ZWICKER

Abstract. Consider an election in which each of the n voters casts a vote consisting of a strict preference ranking of the three candidates A, B, and C. In the limit as $n \to \infty$, which scoring rule maximizes, under the assumption of Impartial Anonymous Culture (uniform probability distribution over profiles), the probability that the Condorcet candidate wins the election, given that a Condorcet candidate exists? We produce an analytic solution, which is not the Borda Count. Our result agrees with recent numerical results from two independent studies, and contradicts a published result of Van Newenhizen (1992).

1. Introduction

We wish to consider elections in which n voters select a winner from among the three candidates A, B, and C. We assume that each voter expresses, as his or her vote, a strict, complete, and transitive preference ranking of the candidates; in particular, no voter expresses indifference between any two candidates. Each voter chooses, then, from among six possible rankings:

$$\begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\ \\ A & C & C & B & B & A \\ B & B & A & A & C & C \\ C & A & B & C & A & B \end{array}$$

Here, each n_i is equal to the number of voters who express the associated ranking. Thus, any 6-tuple $P = (n_1, n_2, n_3, n_4, n_5, n_6)$ of non-negative integers that sum to n tells us how many voters chose each of the rankings in a given election. Such a tuple is known as a *profile*.

Numerous criteria have been suggested to help select which voting systems, among various systems for choosing winners from profiles, best reflect the cumulative will of the electorate. One of the most common of these is the *Condorcet Criterion*. If U and V are candidates, let $U >^C V$ indicate that candidate U defeats V in the pairwise majority election between these two (strictly more voters ranked U over V than ranked V over U); we'll use $U \geq^C V$ for the corresponding weak relation. A candidate is the *Condorcet winner* if she defeats each other candidate in pairwise majority elections.

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It is well known [3, 11] that a Condorcet winner need not exist. However, many feel that a reasonable voting system should elect the Condorcet winner whenever such a candidate exists; this is the *Condorcet Winner Criterion*.

The cost of completely meeting this criterion can be high, however, because it forces us to sacrifice certain other desirable requirements [8]. So it has become common to consider, as a measure of partial fulfillment, the *Condorcet efficiency* of a voting system S, which is the conditional probability that S elects the Condorcet winner, given that a Condorcet winner exists.

Condorcet efficiency depends, of course, on the underlying probability distribution describing the likelihood that various profiles are observed. Many such distributions have been considered, but the two most common assumptions are:

> *Impartial Culture (IC)*: voters choose a preference ranking randomly and independently with probability 1/6 of choosing any particular ranking.
>
> *Impartial Anonymous Culture (IAC)*: each possible preference profile is equally likely.[1]

Further discussion of these two assumptions may be found in Berg [1], Gehrlein [4], and Stensholt [12].

In the general discussion that follows, we will assume that the underlying probability distribution is *neutral* — that is, it treats the candidates symmetrically.[2] Both IAC and IC are neutral.

Our focus is on the Condorcet efficiency of *scoring rules* (also called *weighted scoring rules*). For three candidates, a scoring rule is determined by a vector $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of real-number scoring weights satisfying both $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and $\alpha_1 < \alpha_3$. Each voter awards α_1 points to her bottom-ranked candidate, α_2 points to her second-ranked candidate, and α_3 points to her most favored candidate. The winner is the candidate with the highest number of total points, and in fact the point totals determine a complete ranking of the candidates (with ties possible).

It is has long been known that any order-preserving affine transform

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \langle a\alpha_1 + b, a\alpha_2 + b, a\alpha_3 + b \rangle$$

[1] To appreciate the difference between these two assumptions, note that under IC the probability distribution over profiles is not uniform; it has a peak at the central point $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$. As n increases, more and more of the "area" under the "curve" is concentrated very close to the central point, in such a way that the distribution approaches a "spherically" symmetric one with a huge spike at the center. (We use quotation marks here, because all the action is of course in a higher dimension.)

[2] Specifically, we assume that each permutation of the candidates leaves fixed the probability of any event described in terms of specific candidates.

(where we require $a > 0$ and b to be real-number constants) produces a second vector of scoring weights that induces the identical voting system; for every profile, total score yields the same ranking of candidates using one vector as it does using its transform. Thus, we can normalize any vector so that $\alpha_1 = -1$ and $\alpha_3 = 1$. Under this normalization (others are possible) the voting system is completely determined by the middle scoring weight, which we term β and which satisfies $-1 \leq \beta \leq 1$. Two scoring systems are particularly worth noting in this connection: plurality voting ($\beta = -1$) and the Borda count ($\beta = 0$).

Let us introduce a bit of notation here. Suppose for a moment that our profile P is fixed, and U and V are candidates. We'll use U^β to denote the total of all points awarded to U using the scoring weights $\langle -1, \beta, 1\rangle$, and we will write $U >^\beta V$ to mean $U^\beta > V^\beta$ and $U \geq^\beta V$ to mean $U^\beta \geq V^\beta$. We are now ready to state the three questions that most naturally suggest themselves for study, given our line of inquiry.

Question #1. *Which value of β maximizes the probability that $A >^\beta B$, given that $A >^C B$?*[3]

(Interpretation: Which value of β maximizes the probability, for any two candidates, that the scoring system will rank them in the same order as pairwise majority?)

Question #2. *Which value of β maximizes the probability that $A >^\beta B$ and $A >^\beta C$, given that $A >^C B$ and $A >^C C$?*

(Interpretation: Which value of β maximizes the probability that the scoring winner will equal the Condorcet winner, given that a Condorcet winner exists?)

Question #3. *Which value of β maximizes the probability that $A >^\beta B >^\beta C$, given that $A >^C B >^C C$ and $A >^C C$?* [4]

(Interpretation: Which value of β maximizes the probability that the scoring ranking will equal the Condorcet ranking, given that a Condorcet ranking exists?)

These interpretations may seem, at first, to be too general, but note (for example, in question 2) that neutrality implies that the probability that a

[3]Here, as elsewhere, we are somewhat arbitrarily favoring strict inequalities over weaker ones. This choice is of little account, because for any reasonable probability distribution, as $n \to \infty$ the probability of a tie in either type of ordering (such as $A \geq^C B$ and $B \geq^C A$) becomes vanishingly small. Discounting ties becomes automatic when we later reduce the problem to one of calculating volumes of polytopes, because the volume of such a region is the same regardless of whether the region includes the appropriate sections of its bounding hyperplanes or not, and it is exactly the points lying on these bounding hyperplanes that correspond to ties.

[4]The last inequality is needed because $>^C$ may be intransitive; it would be redundant for $>^\beta$.

particular candidate is the scoring winner, given that that candidate is the Condorcet winner, is equal to the probability that the scoring winner is the Condorcet winner, given that a Condorcet winner exists.

Question 2 is the most important to the current study. Van Newenhizen [14] considered all three questions in an attempt to show that Borda Rule maximized all three of the associated probabilities under a family of probability distributions for voter profiles, including IAC.

Question #4. *Which value of β maximizes the probability that $A >^\beta C >^\beta B$, given that $A >^C B >^C C$ and $A >^C C$?*

The importance of question 4 is not obvious. As we will see, it helps to complete a package that makes the answer to question 2 seem more reasonable and, with the benefit of hindsight, almost predictable.

Gehrlein and Fishburn [6] consider question 2 with the assumption of IC. The results of that study found the range of β values to maximize the associated probability for small n, to show that Borda Rule was not included in the range of those weights. However, as $n \to \infty$ under IC, it was shown that Borda Rule uniquely maximized the probability from question 2. Tataru and Merlin [13] use the assumption of IC as $n \to \infty$ to develop a representation for the probability that $B >^\beta A$ and $C >^\beta A$, given that $A >^C B$ and $A >^C C$, and show that it is the same as the representation for the probability that $A >^\beta B$ and $A >^\beta C$, given that $B >^C A$ and $C >^C A$. Their study begins with notions of geometric models of voting phenomena, as developed by Saari and Tataru [9], and then obtains probability representations by applying results of Schläfli [10].

Gehrlein [5] addresses the probability in question 2 for small n under the assumption of IAC to find the range of β values to maximize that probability. Results do not suggest that Borda Rule will maximize the probability as $n \to \infty$ with IAC, as it did with IC. Lepelley, Pierron and Valognes [7] also present computational results for large n to suggest that Borda Rule does not maximize the probability from question 2 as $n \to \infty$ with IAC. These observations are not in agreement with the results given in Van Newenhizen [14], as mentioned above. The current study considers the limiting case as $n \to \infty$ with the assumption of IAC, and develops closed-form representations for the probabilities, to resolve the inconsistencies in these observations.

The rest of this paper is organized as follows. In section 2 we explain how the problem may be reduced to one of pure geometry. Our main results are detailed answers to questions 1, 2, 3, and 4, that are stated without proof in section 3. Some subtleties of the problem, which seem to make the IAC case somewhat more complex than the IC case, are discussed in section 4. In section 5, we discuss the role of question 4 and of certain symmetry arguments in providing some intuition for the main results. It seems, however, that these ideas are unlikely to yield short proofs that substitute fully for the detailed geometrical analyses and proofs, which appear in section 6.

2. Geometrization

If we divide each of the vote counts, n_i, that appears in our vote profile $P = (n_1, n_2, n_3, n_4, n_5, n_6)$ by the number of votes, n, we obtain a normalized profile $\bar{P} = (x_1, x_2, x_3, x_4, x_5, x_6)$ of numbers $x_i = n_i/n$ that represent the fractions of voters who favor each of the six strict preference rankings. Each of the relationships we require can be expressed as easily in terms of the x_i as in terms of the n_i. For example, $A >^C B$ if, and only if,

$$x_1 - x_2 + x_3 - x_4 - x_5 + x_6 > 0 \tag{H_{1a}}$$

and $A >^\beta B$ if, and only if,

$$(1-\beta)x_1 - (1+\beta)x_2 + (1+\beta)x_3 - (1-\beta)x_4 + 2(-x_5 + x_6) > 0. \tag{H_{1b}}$$

Note that each $x_i \geq 0$ and $\sum x_i = 1$, so the normalized profiles correspond to those points in Δ^5, the 5-simplex in $\mathbf{R}^6$, for which each coordinate is a rational number that may be written with denominator n. These points form a regular lattice in Δ^5, in which each point spaced away from the boundary has $6 \times 5 = 30$ nearest neighbors. Under IAC, our probability measure puts equal weight on each of these lattice points inside Δ^5. In figure 1 we have sketched the analogous situation for the 3-simplex in $\mathbf{R}^3$, showing first how Δ^3 sits inside $\mathbf{R}^3$ and then illustrating the lattice for the case $n = 5$ (for which each point spaced away from the boundary has $3 \times 2 = 6$ nearest neighbors.)

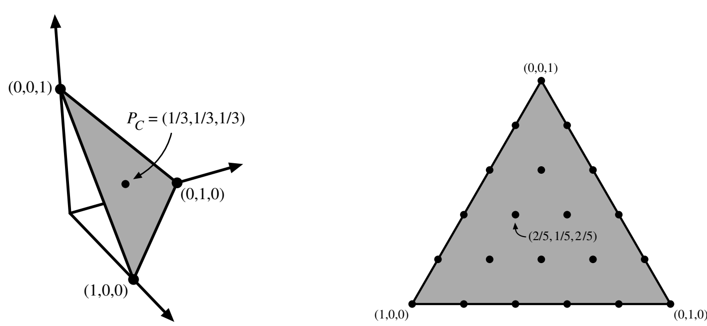

Figure 1. The 2-simplex, Δ^2, in three-space, containing the central point $P_C = (1/3, 1/3, 1/3)$ is shown on the left. On the right we show the lattice points (for $n = 5$) within this simplex.

The entire region Δ^5 in $\mathbf{R}^6$ is the graph of the solution set of the following system T_0 of linear equations and inequalities:

$$\begin{gathered} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0, \text{ and } x_6 \geq 0. \end{gathered} \tag{T_0}$$

Now consider the region we shall call R_1 (because it is the principal region of interest for question 1), defined as the graph of the solution set of the system T_1 consisting of T_0 together with inequalities H_{1a} and H_{1b}, and also consider region R_{1a} similarly defined in terms of the system T_{1a} consisting of T_0 with only inequality H_{1a} added in. It is evident that the probability for question 1 (of our scoring system ranking of two candidates being in agreement with the pairwise majority outcome) is given by the following expression:

$$\lim_{n\to\infty} \frac{\text{number of lattice points in region } R_1}{\text{number of lattice points in region } R_{1a}}$$

Because our regions are not complicated (they are solution sets of a linear system) this limit agrees with[5] the ratio

$$\frac{\text{volume of region } R_1}{\text{volume of region } R_{1a}} \tag{1}$$

where by volume we intend "5-volume".

Furthermore, changes in β do not affect the volume of region R_{1a}, so the value of β maximizing ratio (1) is the same as that maximizing the volume of R_1. Thus, our original question 1 can be translated as follows:

Question #1 [volume equivalent form]**.** *What value of β maximizes the volume of region R_1?*

We can say more about these inequalities. Let M_Δ be the hyperplane containing Δ^5, which has equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$. We will primarily be concerned with what goes on within this plane. Consider the equality corresponding to inequality H_{1a}:

$$x_1 - x_2 + x_3 - x_4 - x_5 + x_6 = 0. \tag{P_{1a}}$$

This hyperplane, P_{1a}, intersected with M_Δ, forms a hyperplane M_{1a}. Note that M_{1a} passes through the central point P_C of Δ^5, and that the solution of the original inequality, H_{1a}, (again, taken within M_Δ) forms a half-space within M_Δ. (For notational purposes, if we have a half-space in six-space given by the inequality H_x, its bounding hyperplane will be the equality denoted P_x, which intersects with M_Δ to form a 5-dimensional hyperplane, M_x.) The equality corresponding to inequality H_{1b},

$$(1-\beta)x_1 - (1+\beta)x_2 + (1+\beta)x_3 - (1-\beta)x_4 + 2(-x_5 + x_6) = 0, \tag{P_{1b}}$$

yields a second hyperplane $M_{1b}(\beta)$ that also passes through the central point P_C of Δ^5. Thus R_1 consists of a "wedge" cut out of Δ^5 by a pair of hyperplanes that pass through the middle of Δ^5.

The first question can be rephrased informally, then, as follows:

[5] The necessary background for a proof of this assertion is given in [2].

Question #1 [informal version]. *As β varies from -1 to $+1$, the plane $M_{1b}(\beta)$ swivels about the central point of the simplex. For which β does this plane, together with the fixed plane M_{1a}, cut the largest wedge R_1 out of Δ^5?*

A schematic depiction, using Δ^2 in place of Δ^5, appears as figure 2.

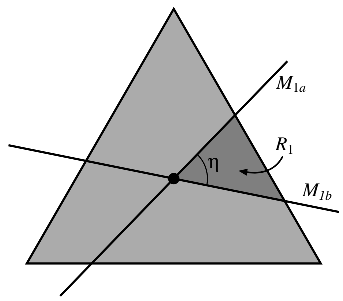

FIGURE 2. A fixed line M_{1a} and a moving line M_{1b} cut a region R_1 from the 2-simplex that is largest when the angle η between the lines is largest.

Each of the other questions can be similarly expressed. The translation of the second question is as follows:

Question #2 [informal version]. *As β varies from -1 to $+1$, the planes $M_{2b_1}(\beta)$ and $M_{2b_2}(\beta)$ both swivel about the central point of the simplex. For which β do these two planes, together with the two fixed planes M_{2a_1} and M_{2a_2}, cut the largest wedge R_2 out of Δ^5?*

The equations of the half-spaces (in six-space that induce these planes in M_Δ) together with the associated inequalities in terms of $>^C$ and $>^\beta$, are as follow:

$$(A >^C B) \quad x_1 - x_2 + x_3 - x_4 - x_5 + x_6 > 0 \tag{H_{2a_1}}$$

$$(A >^C C) \quad x_1 - x_2 - x_3 + x_4 - x_5 + x_6 > 0 \tag{H_{2a_2}}$$

$$(A >^\beta B) \quad (1-\beta)x_1 - (1+\beta)x_2 + (1+\beta)x_3 - (1-\beta)x_4 + 2(-x_5 + x_6) > 0 \tag{H_{2b_1}}$$

$$(A >^\beta C) \quad 2(x_1 - x_2) - (1-\beta)x_3 + (1+\beta)x_4 - (1+\beta)x_5 + (1-\beta)x_6 > 0 \tag{H_{2b_2}}$$

Each of the questions 3 and 4 give rise to three fixed hyperplanes (from the inequalities $A >^C B$, $B >^C C$, and $A >^C C$) and two moving hyperplanes (from $A >^\beta B$ and $B >^\beta C$, for question 3, and from $A >^\beta C$ and $C >^\beta B$ for question 4). The equations for these are given in section 6.

3. Statements of Main Results

Theorem 1. *Consider elections using scoring systems for which there are three candidates and n voters, and for which a vote consists of a strict preference ranking of the candidates. Assume that all profiles are equally likely (IAC). Then for any two candidates, the limit, as $n \to \infty$, of the value of β maximizing the probability that the scoring system ranks these two in the same order as they are ranked by pairwise majority is 0; that is, the Borda count maximizes the limiting probability in this context. This number is the unique value of β in $[-1, 1]$ for which the function F_1 given below satisfies $F_1'(\beta) = 0$.*

The function $F_1(\beta)$ defined on the interval $[-1, 1]$ whose value is the limiting conditional probability described above (i.e., the ratio of the 5-volume of region R_1 to that of region R_{1a}) is given by the following formula:

$$F_1(\beta) = \begin{cases} \tilde{F}_1(-\beta)/V_1 & \text{for } -1 \le \beta \le 0 \\ \tilde{F}_1(\beta)/V_1 & \text{for } 0 \le \beta \le 1 \end{cases}$$

where

$$\tilde{F}_1(\beta) = \frac{\sqrt{6}}{3840} \cdot \frac{131 + 131\beta - 47\beta^2 - 11\beta^3 + 4\beta^4}{(1+\beta)(3-\beta)(3+\beta)}$$

and $V_1 = \sqrt{6}/240$ is the volume of region R_{1a}. This function is symmetric about the y axis ($\beta = 0$), and its graph appears as figure 3. Although the formula presented here was computed using the method described in section 6, it was duplicated independently by the second-named author using an algebraic partitioning of the space enclosed by the hyperplanes.

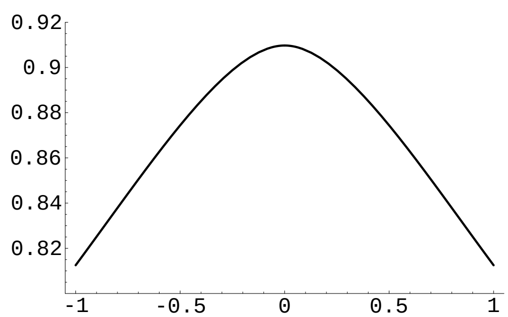

Figure 3. The graph of the function $F_1(\beta)$, which gives the conditional probability of question 1. It is symmetric about the y axis and has a unique maximum at $\beta = 0$.

Theorem 2. *Consider elections using scoring systems for which there are three candidates and n voters, and for which a vote consists of a strict preference ranking of the candidates. Assume that all profiles are equally likely (IAC). Then the limit, as $n \to \infty$, of the value of β maximizing the probability that the scoring winner of the election is the Condorcet winner, given* *that a Condorcet winner exists, is* $\beta = -0.25544$ *(which is correct to five decimal places); in particular, the Borda count does not maximize the limiting probability in this context. This number is the unique value of* β *in* $[-1,1]$ *for which the function* F_2 *given below satisfies* $F_2'(\beta) = 0$.

The function $F_2(\beta)$ defined on the interval $[-1,1]$ whose value is the limiting conditional probability described above (i.e., the ratio of the 5-volumes of the corresponding regions R_2 and R_{2a}) is given by the following formula:

$$F_2(\beta) = \begin{cases} F_2^-(\beta)/V_2 & \text{for } -1 \le \beta \le 0 \\ F_2^+(\beta)/V_2 & \text{for } 0 \le \beta \le 1 \end{cases}$$

where

$$F_2^-(\beta) = \frac{\sqrt{6}}{155{,}520}\left(\frac{3321 - 10449\beta + 9558\beta^2 - 1152\beta^3}{(1-\beta)^3(9-\beta^2)} + \frac{-1467\beta^4 + 191\beta^5 + 28\beta^6 + 2\beta^7}{(1-\beta)^3(9-\beta^2)}\right)$$

and

$$F_2^+(\beta) = \frac{\sqrt{6}}{155{,}520}\left(\frac{3321 + 19440\beta + 35073\beta^2 + 18810\beta^3 - 7083\beta^4}{(1+\beta)^3(9-\beta^2)(1+3\beta)} + \frac{-6518\beta^5 + 1381\beta^6 + 844\beta^7 + 12\beta^8}{(1+\beta)^3(9-\beta^2)(1+3\beta)}\right)$$

and $V_2 = \sqrt{6}/384$ is the volume of the region R_{2a} that is the intersection of the simplex Δ^5 and the two fixed hyperplanes H_{2a_1} and H_{2a_2}. This function is not symmetric about the y axis ($\beta = 0$), nor is it symmetric about the line $\beta = -0.25544$. Its graph appears as figure 4.

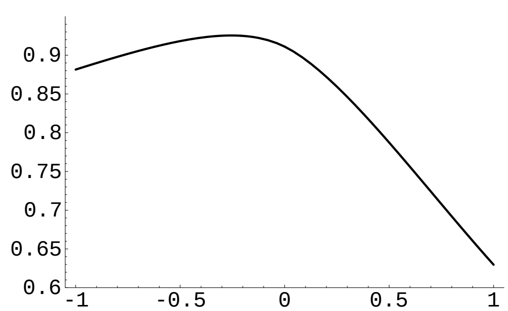

FIGURE 4. The graph of the function $F_2(\beta)$, which gives the conditional probability of question 2. It is not symmetric and has a maximum at approximately $\beta = -0.25544$.

Theorem 3. *Consider elections using scoring systems for which there are three candidates and n voters, and for which a vote consists of a strict preference ranking of the candidates. Assume that all profiles are equally likely (IAC). Then the limit, as $n \to \infty$, of the value of β maximizing the probability that the ranking of candidates by total score agrees with the transitive ranking by pairwise majorities, given that the pairwise majorities do yield a transitive ranking, is 0; that is, the Borda count maximizes the limiting probability in this context. Zero is the unique value of β in $[-1,1]$ for which the function F_3 given below satisfies $F_3'(\beta) = 0$.*

The function $F_3(\beta)$ defined on the interval $[-1,1]$ whose value is the limiting conditional probability described above (i.e., the ratio of the 5-volumes of the corresponding regions R_3 and R_{3a}) is given by the following formula:

$$F_3(\beta) = \begin{cases} \tilde{F}_3(-\beta)/V_3 & \text{for } -1 \le \beta \le 0 \\ \tilde{F}_3(\beta)/V_3 & \text{for } 0 \le \beta \le 1 \end{cases}$$

where

$$\tilde{F}_3(\beta) = \frac{\sqrt{6}}{311{,}040}\left(\frac{2997 + 17982\beta + 32643\beta^2 + 16074\beta^3 - 7767\beta^4}{(1+\beta)^3(9-\beta^2)(1+3\beta)}\right.$$
$$\left. + \frac{-5926\beta^5 + 1241\beta^6 + 926\beta^7 + 6\beta^8}{(1+\beta)^3(9-\beta^2)(1+3\beta)}\right)$$

where $V_3 = \sqrt{6}/768$. This function is symmetric about the y axis ($\beta = 0$), and its graph appears as figure 5.

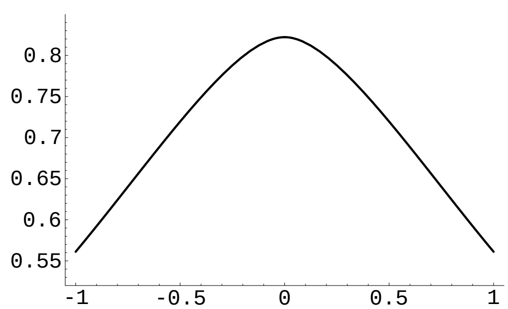

FIGURE 5. The graph of the function $F_3(\beta)$, which gives the conditional probability of question 3. It is symmetric about the y axis and has a unique maximum at $\beta = 0$.

Theorem 4. *Consider elections using scoring systems for which there are three candidates and n voters, and for which a vote consists of a strict preference ranking of the candidates. Assume that all profiles are equally likely (IAC). Then the limit, as $n \to \infty$, of the value of β maximizing the probability that the ranking of candidates by total score agrees, in terms of the top candidate, with the transitive ranking by pairwise majorities, but disagrees* *in terms of the ordering of the second and third ranked candidates, given that the pairwise majorities do yield a transitive ranking, is* -1*; that is, plurality voting maximizes the limiting probability in this context.*

The function $F_4(\beta)$ defined on the interval $[-1, 1]$ whose value is the limiting conditional probability described above (i.e., the ratio of the 5-volumes of the corresponding regions R_4 and R_{4a}) is given by the following formula:

$$F_4(\beta) = \begin{cases} F_4^-(\beta)/V_4 & \text{for } -1 \le \beta \le 0 \\ F_4^+(\beta)/V_4 & \text{for } 0 \le \beta \le 1 \end{cases}$$

where

$$F_4^-(\beta) = \frac{\sqrt{6}}{155{,}520}\left(\frac{162 - 1215\beta + 4131\beta^2 - 6876\beta^3 + 4878\beta^4}{(1-\beta)^3(9-\beta^2)(1-3\beta)} + \frac{-667\beta^5 - 893\beta^6 + 422\beta^7 - 6\beta^8}{(1-\beta)^3(9-\beta^2)(1-3\beta)}\right)$$

and

$$F_4^+(\beta) = \frac{\sqrt{6}}{155{,}520}\left(\frac{162 + 729\beta + 1215\beta^2 + 1368\beta^3 + 342\beta^4}{(1+\beta)^3(9-\beta^2)(1+3\beta)} + \frac{-296\beta^5 + 70\beta^6 - 41\beta^7 + 3\beta^8}{(1+\beta)^3(9-\beta^2)(1+3\beta)}\right)$$

and $V_4 = \sqrt{6}/768$. The graph of $F_4(\beta)$ is decreasing on the interval from -1 to 0.63311 and increasing on the interval from 0.63311 to 1. This function is not symmetric about the y axis ($\beta = 0$), nor is it symmetric about the line $\beta = 0.63311$. Its graph appears as figure 6.

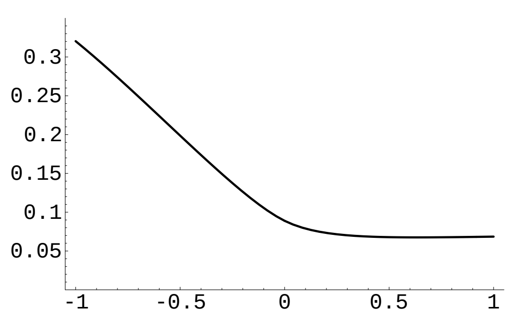

FIGURE 6. The graph of the function $F_4(\beta)$, which gives the conditional probability of question 4. It is not symmetric and has a maximum at $\beta = -1$.

4. The Arguments in Van Newenhizen [1992]

Much of the intuition behind our results arises from an understanding of what goes wrong in the approach of Van Newenhizen [14], here termed the *basic approach*, which is the natural one that might first suggest itself. Because question 1 entails only one fixed plane and one moving plane, we begin by considering this question. The basic approach considers the angle η between the two hyperplanes M_{1a} and $M_{1b}(\beta)$ (see figure 2), shows that when $\beta = 0$ the angle η is maximized, and concludes that when $\beta = 0$ the volume of the resulting wedge R_1 is also maximized.

Figure 2 certainly suggests that a larger value of η yields a wedge R_1 that strictly contains (as a set of points) the version of R_1 arising from a smaller η value. However, figure 2 is a schematic representation of a more complex situation in a higher dimensional space. The two hyperplanes M_{1a} and $M_{1b}(\beta)$ each contain the central point P_C, so their intersection contains P_C as well. In figure 2 it appears as if P_C is the only point in this intersection, from which it would follow that this intersection is fixed as β varies. In fact, the intersection, inside M_Δ, of these two 4-dimensional hyperplanes is itself a 3-dimensional object that swivels about P_C as β varies. Consequently the R_1 for some β value corresponding to a certain angle η is likely to neither contain nor be contained by the R_1 for a different β value corresponding to a larger or smaller angle η. As β varies, some points are added to R_1 and other points are removed, as we show in section 6.4.

Suppose, however, that the wedge R_1 were cut from a ball (solid sphere), rather than from a simplex, by a pair of hyperplanes passing through the center of the sphere. Then thanks to spherical symmetry, the largest volume would necessarily correspond to the largest angle η between the planes, regardless of whether or not each R_1 with a larger η contained as a subset each R_1 with a smaller η. For the actual case at hand, however, we are faced with the following plausible scenario: perhaps as β varies it happens that while η is decreasing the position of R_1 is swinging around in such a way that the portion of the wall of our simplex that forms part of the boundary of R_1 is becoming more distant from the central point.

Let us sum up. The probability distribution under consideration is not spherically symmetric. Hence, it does not follow that the largest η yields the largest volume, unless one can show that the intersection of the hyperplanes is fixed as β varies. But this intersection is not fixed, so there is a gap in the basic approach. This same flaw is inherent in the application of these methods to questions #2 and #3, as well.

In the case of questions #1 and #3, the result obtained by the basic approach turns out to be correct, so conceivably there is some way to patch the argument, rather than starting from scratch. This seems as if it would be difficult to do. In the case of question 2, the result obtained from the basic approach turns out to be incorrect. Could the flaw discussed in the previous paragraph explain why this approach yields the wrong answer to question 2?

The answer is not immediately apparent. Because question 2 (as well as #3) entails more than two planes, a second type of difficulty arises. To focus on this second issue, let us temporarily move the first issue off the table by ignoring the simplex (and all its bounding hyperplanes), and pretending instead that our probability distribution is the uniform distribution, inside M_Δ, over the solid ball centered at the point P_C that is common to all remaining hyperplanes. The basic approach considers two of these angles and shows that these are each maximized when $\beta = 0$. But as there are several planes and more than two angles involved, the subtleties of higher dimensional geometry are such that it is not immediately clear which condition on these angles is the appropriate one, given our temporary assumption of spherical symmetry.

Does $\beta = 0$ maximize the spherical variant of region R_2? To answer this, we look first at a three-dimensional analogue: two planes cutting a wedge out of a three-ball. If the planes pass through the center of the ball, their intersection forms a line, L, passing through antipodal points on the sphere, and the wedge is in the shape of an "orange slice". The volume of this slice is maximized when the area of the piece of "orange peel" on the slice is maximized. This area is in the form of a spherical wedge known as a *lune*. Note that planes perpendicular to the line L cut the lune in circular arcs, with the longest arc for the plane through the center of the sphere (i.e., an "equatorial" arc). The lune of greatest area will be achieved by the two planes that form the longest equatorial arc.

The corresponding higher-dimensional situation has four hyperplanes passing through the center of the 5-ball in the five-dimensional space M_Δ. They intersect in a line, L, and cut a five-dimensional wedge from the 5-ball. This wedge is maximized when the corresponding four-dimensional lune cut from the 4-sphere is maximized. The hyperplanes perpendicular to L slice the four-dimensional lune in three-dimensional spherical regions, in this case, spherical tetrahedra whose four faces correspond to the four original hyperplanes. Again, the largest tetrahedron is the equatorial one, and the volume of the five-dimensional wedge is maximized when the three-dimensional volume of this equatorial tetrahedron is greatest.

The basic approach argues that the greatest volume occurs for $\beta = 0$ by looking at one angle between two of the four slicing hyperplanes and a second between the other two hyperplanes, but this does not seem sufficient justification for the claim. On the other hand, it turns out that for *any* two of the four hyperplanes, their maximal angle occurs when $\beta = 0$. Now the angles between the hyperplanes are actually the angles between the faces of the spherical tetrahedron, and it seems reasonable that the volume of this tetrahedron would be largest when its angles are largest. Thus the maximal volume for the spherical R_2 region does appear to be at $\beta = 0$.

The result is that although the reasoning is not sufficient, the conclusion reached by the basic approach is nonetheless correct in the case of a spherically symmetric region. Thus the original issue discussed above (namely that the region is not spherically symmetric, and so a larger wedge does not necessarily cut a larger volume) seems to be the fatal flaw in the basic approach.

5. Symmetry and the Role of Question #4

The detailed geometrical analysis in section 6 proves that, in the context of this paper, the Borda count is not the most Condorcet efficient among scoring systems, yet it is the "best" scoring system in the sense of questions #1 and #3. However, the material in section 6 does not provide a short and intuitively satisfying "story" explaining why things turn out this way, or why the value of β is closer to -1 than to 1 in the most Condorcet efficient scoring system. Does such a story exist? The answer seems to be "Yes, to a limited extent."

We begin with a short and self-contained argument explaining why the functions $F_1(\beta)$ and $F_3(\beta)$ must be symmetric about the line $\beta = 0$. (This argument does not rest on the formulae for these functions, which were derived through the longer geometrical analysis.) We discuss why the same argument does not apply to $F_2(\beta)$. Of course, symmetry does not show that the maximum is located at $\beta = 0$ (because there might exist several maxima located symmetrically about the line $\beta = 0$). Then we turn to the role of question 4 in providing a plausibility argument for why the value of β is closer to -1 than to 1 in the most Condorcet efficient scoring system.

Theorem 5. *The functions $F_1(\beta)$ and $F_3(\beta)$ are each symmetric about the line $\beta = 0$.*

Proof. We'll show $F_3(\beta)$ is symmetric and leave the proof for $F_1(\beta)$ to the reader. Consider the probability p_1 that

$$A >^{\beta} B \quad \text{and} \quad B >^{\beta} C \tag{2}$$

given that

$$A >^{C} B, \quad B >^{C} C \quad \text{and} \quad A >^{C} C. \tag{3}$$

We'll show that p_1 is equal to the probability p_2 that

$$C >^{-\beta} B \quad \text{and} \quad B >^{-\beta} A, \tag{4}$$

given that

$$C >^{C} B, \quad B >^{C} A \quad \text{and} \quad C >^{C} A. \tag{5}$$

For reasons we have already discussed, $p_1 = F_3(\beta)$ and $p_2 = F_3(-\beta)$, so it will follow that $F_3(\beta) = F_3(-\beta)$, as desired. Now we have

$$p_1 = \frac{\text{Volume of Region } R_3}{\text{Volume of Region } R_{3a}}$$

and

$$p_2 = \frac{\text{Volume of Region } R_3^*}{\text{Volume of Region } R_{3a}^*}$$

where region R_{3a} is defined by the inequalities in T_0 (for the simplex) together with (3), while for R_3 we add (2) as well. Similarly, region R_{3a}^* is defined by the inequalities in T_0 (for the simplex) together with (5), while for R_3^* we add (4) as well.

Next, consider the function $G\colon \Delta^5 \to \Delta^5$ by

$$G(x_1, x_2, x_3, x_4, x_5, x_6) = (x_2, x_1, x_4, x_3, x_6, x_5).$$

As a map from Δ^5 to Δ^5, G is clearly a volume-preserving bijection, so it follows that

$$p_1 = \frac{\text{Volume of Region } G(R_3)}{\text{Volume of Region } G(R_{3a})}$$

The regions $G(R_3)$ and $G(R_{3a})$ are described by the inequalities obtained by transforming the earlier inequalities (2) and (3) via G. It remains only, then, to show that these transformed inequalities are (4) and (5), and it will follow that $G(R_3) = R_3^*$ and $G(R_{3a}) = R_{3a}^*$, whence $p_1 = p_2$.

As the effect of applying G to a profile is to have each voter flip his or her preference ranking upside down, it is immediate that applying G to any inequality of form $U >^C V$ transforms it into the inequality $V >^C U$. Also, the total score $U^\beta(P)$ for a candidate U under a profile P is easily seen to satisfy that $U^\beta(P) = -U^{-\beta}(G(P))$ from which it follows that applying G to any inequality of form $U >^\beta V$ transforms it into the inequality $V >^{-\beta} U$. Now it is immediate that G transforms (2) into (4) and (3) into (5), which completes the proof. $\square$

It is interesting to observe what happens when the above argument is applied to question 2, asking which value of β yields the most Condorcet efficient scoring system, or equivalently, which value of β maximizes the probability that $A >^\beta B$ and $A >^\beta C$, given that $A >^C B$ and $A >^C C$. Applying the same transform G to the question tells us that the question 2 answer is the same as the value of β maximizing the probability that $B >^{-\beta} A$ and $C >^{-\beta} A$, given that $B >^C A$ and $C >^C A$. In this case, the transformed question is not equivalent to the original: it asks us which value of $-\beta$ maximizes the probability that the scoring *loser* is the Condorcet *loser*, given that a Condorcet loser exists.

We cannot conclude that $F_2(\beta)$ is symmetric. But the argument does yield some information when thus applied — it tells us that if we were to plot

"Condorcet loser efficiency" as a function of β, its graph would be the reflection across the line $\beta = 0$ of the graph of $F_2(\beta)$.

We turn next to a discussion of theorem 2. The most surprising result in this paper is certainly the fact that the Borda count ($\beta = 0$) is not most Condorcet efficient among scoring systems. Should we have expected this, and might we have guessed that efficiency is maximized with a negative β value? We offer a qualified "yes," one certainly aided by hindsight, in what follows.

Assume that A is the Condorcet candidate in an election with three candidates, so that $A >^C B$ and $A >^C C$. Then there is no cycle, so there exists a Condorcet ranking. Ignoring the possibility of ties (see footnote 1), either

Case 1: $A >^C B >^C C$ with $A >^C C$, or
Case 2: $A >^C C >^C B$ with $A >^C B$.

By symmetry, the probabilities of these two cases holding are equal. We will work on case 1; our reasoning applies just as well for case 2.

The desired outcome — that A is the scoring winner with $A >^\beta B$ and $A >^\beta C$ — can happen in two ways. Either

Case 1.1: $A >^C B >^C C$ (with $A >^C C$) and $A >^\beta B >^\beta C$, or
Case 1.2: $A >^C B >^C C$ (with $A >^C C$) and $A >^\beta C >^\beta B$.

Given that we are in case 1, the probability that case 1.1 holds is maximized when $\beta = 0$ (Borda count); this is precisely what theorem 3 states.

Theorem 4 tells us that the value of β maximizing the probability of case 1.2, given that case 1 holds, is $\beta = -1$ (plurality voting). We will return to theorem 4 in a moment. Now the probability that case 1 occurs (equivalently, the volume of a certain 5-dimensional region) is equal to the sum of the probabilities for the exclusive cases 1.1 and 1.2 (equivalently, the case 1 region is partitioned into the disjoint case 1.1 and 1.2 subregions, so that its volume is the sum of the volumes of these subregions). Given that one probability (or subregion volume) is maximized when $\beta = 0$ and the other is maximized when $\beta = -1$, it seems reasonable that the sum might be maximized by some compromise value of β between -1 and 0.

We did wonder whether it is possible to bypass the computational geometry of section 6 by making the above argument rigorous. However, the directions that we explored seem to point towards showing that the volume of the region for question 4 steadily decreases as β varies from -1 to $+1$. This will not be possible to show, since, in fact, F_4 is increasing for $\beta > 0.63311$, as illustrated in figure 7.

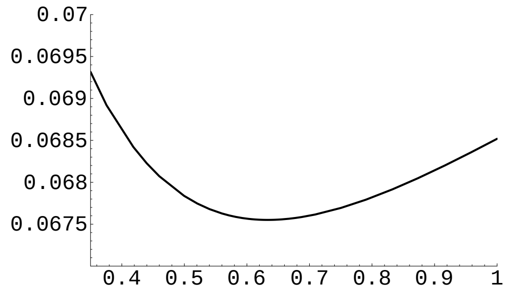

FIGURE 7. The graph of $F_4(\beta)$ appears to be decreasing, but there is a small rise to the right of $\beta = 0.63311$, shown here with an exaggerated scale on the vertical axis.

6. Proofs of Theorems 1–4

In this section, we give the details of the calculations that lead to the formulae for the volumes given in section 3. The method used is described in general terms in section 6.1. To carry out the process, the combinatorial structure of the region whose volume is to be computed must be determined; this is done in detail in section 6.2 for the region R_1 associated with theorem 1. This result is used in section 6.3 to compute the volume of R_1, and the fact that the maximum volume occurs at $\beta = 0$ is verified in section 6.4. The data needed to perform similar computations for the regions R_2, R_3 and R_4 are given in section 6.5, but the details are left to the reader.

6.1. **The Volume of a Convex Region.** One way to determine the volume of a region is to break it into smaller pieces, whose volumes can be computed more readily, and then add up the results. In our case, since the region is convex, we can select one of its vertices and decompose the region into a collection of pyramids having this vertex as their apex and the various faces of the region as their bases. We need only consider the faces that don't contain the selected vertex, as the volume will be zero if the base contains the apex of the pyramid. A three dimensional example would be the decomposition of a cube into three congruent square-based pyramids with apexes directly above a corner of their square bases (figure 8); the three pyramids meet along a long diagonal of the cube.

The volume of a pyramid in dimension n is given by $\frac{1}{n}Vh$ where V is the $(n-1)$-dimensional volume of the base, and h is the height of the apex above the base. In two dimensions, this reduces to the usual formula $\frac{1}{2}bh$ for the area of a triangle of base b and height h, and in three dimensions to the formula $\frac{1}{3}Ah$ for the volume of a pyramid (or cone) of height h whose base has area A. To use this formula in general, however, we need to be able to compute the $(n-1)$-volume of the base. In our case, the bases are faces of a convex object and so are themselves convex, hence we can

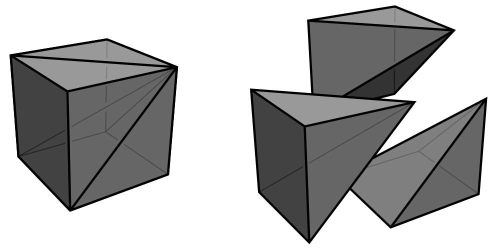

Figure 8. A cube can be divided into three square-based pyramids having a common apex.

compute their volumes by breaking them into pyramids in dimension $n-1$ and adding up the volumes of these pyramids exactly as we did with the original region. This generates a recursive procedure for computing the volume of the region. The base case for the recursion is the 2-dimensional case, where the "pyramid" is a triangle, and the "base" is a line segment; the "volume" of the base is then just the length of this line segment, which can be computed using the distance formula.

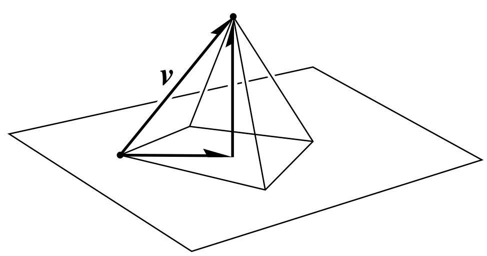

Figure 9. A vector perpendicular to the base can be obtained by subtracting the projection of the vector onto the plane of the base.

To use the volume formula for a pyramid, we also need to know the height of the apex over the hyperplane containing the base. One way to get this is the following: consider the vector $\boldsymbol{v}$, from a point in the hyperplane (say a vertex of the base) to the apex (figure 9); subtract from $\boldsymbol{v}$ the projection of $\boldsymbol{v}$ onto the base hyperplane to obtain a vector perpendicular to the base that points to the apex. The length of this perpendicular vector is the height, h. If we know orthogonal unit vectors that span the base hyperplane, then the projection of $\boldsymbol{v}$ can be computed as the sum of the projections onto the orthogonal basis vectors. Notice that in finding the height, we compute a vector perpendicular to the base hyperplane, so we can extend the orthonormal basis for the base to an orthonormal basis for one dimension higher (i.e., for the space containing the pyramid). This fits in nicely with our recursive procedure, since the pyramid may be part of the base of a higher-dimensional pyramid whose volume we are computing, and we'd need this enlarged basis to get the height of that larger pyramid. (The process is really just Grahm-Schmidt orthonormalization with volume computations mixed in.)

In this way, the volume of a convex polyhedral object in any dimension can be computed effectively, provided the structure of its faces is known. We turn now to our specific region and analyze its shape more carefully.

6.2. **The Structure of the Region $\boldsymbol{R_1}$.** To describe the region we need some terminology. For each $i = 1, \ldots, 6$, define P_i to be the hyperplane $x_i = 0$; recall that M_Δ is the plane $\sum x_i = 1$. Suppose we let $\boldsymbol{u}_i$ be the point in $\mathbf{R}^6$ that has a 1 in the i-th coordinate and 0's everywhere else. Note that the P_i and M_Δ intersect to form a regular 5-simplex in six-space having the $\boldsymbol{u}_i$ as its vertices. (The corresponding object in three-space is the equilateral triangle having vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, lying in the plane $x+y+z=1$ shown in figure 1.) The region R_1 is the intersection of this 5-simplex with two additional halfspaces, H_{1a} and H_{1b} defined in section 2, having P_{1a} and P_{1b} as their bounding hyperplanes

Note that the hyperplane P_{1a} cuts off a section of Δ^5 that does not depend on β, while the hyperplane P_{1b} then cuts off another section that *does* depend on β. Recall that R_{1a} is the result of intersecting Δ^5 with the halfspace H_{1a}, so R_1 is then the result of intersecting R_{1a} with H_{1b}. To compute the volume of R_1, we look more closely at how these intersections occur.

First, let $\boldsymbol{N}_{1a} = \langle 1, -1, 1, -1, -1, 1 \rangle$, a vector normal to the bounding hyperplane, P_{1a}, for H_{1a}. Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ be an arbitrary point in six-space; then $\boldsymbol{x}$ will lie in the halfspace H_{1a} provided $\boldsymbol{N}_{1a} \cdot \boldsymbol{x} \geq 0$. In particular, we can check the six vertices $\boldsymbol{u}_i$ of the 5-simplex to see which vertices are cut off by P_{1a}. Since $\boldsymbol{u}_i$ is all zeros except for a 1 in the i-th coordinate, $\boldsymbol{N}_{1a} \cdot \boldsymbol{u}_i$ is just the i-th coordinate of $\boldsymbol{N}_{1a}$; when this is positive, the vertex remains within the region, otherwise it is cut off. From this, we see that vertices $\boldsymbol{u}_1$, $\boldsymbol{u}_3$, and $\boldsymbol{u}_6$ are in R_{1a}, while $\boldsymbol{u}_2$, $\boldsymbol{u}_4$ and $\boldsymbol{u}_5$ are cut off. Let $S^+ = \{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_6\}$ and $S^- = \{\boldsymbol{u}_2, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

New vertices are formed for R_{1a} when the hyperplane P_{1a} intersects one of the edges of the 5-simplex. This occurs when the two vertices of the edge are on opposite sides of the hyperplane; that is, when one vertex is in the set S^+ and the other is in S^-. Since the 5-simplex has an edge for each pair of its vertices, this means that there are nine new vertices created by slicing the simplex by this hyperplane. We can compute the locations of these vertices by parameterizing the points along the edge with vertices $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ by $\boldsymbol{p}_t = \boldsymbol{p}_0 + t(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ and solving for the t where $\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_t = 0$; i.e., $\boldsymbol{N}_{1a} \cdot (\boldsymbol{p}_0 + t(\boldsymbol{p}_1 - \boldsymbol{p}_0)) = 0$, or $t = \boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0 / (\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0 - \boldsymbol{N}_{1a} \cdot \boldsymbol{p}_1)$. Since $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ each are one of the $\boldsymbol{u}_i$, $\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_i$ is always one of the coordinates of $\boldsymbol{N}_{1a}$, so will be 1 or -1; but since $\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0$ and $\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_1$ must be of opposite sign (the vertices are on opposite sides of the hyperplane), it must be that

$\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0 = -\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_1$. So $t = \boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0 / (\boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0 + \boldsymbol{N}_{1a} \cdot \boldsymbol{p}_0) = 1/2$, no matter which edge we consider. Thus each of the nine edges are cut exactly in half, and the new vertices formed in R_{1a} are

$$\begin{array}{lll} \boldsymbol{v}_{12} = \frac{1}{2}(1,1,0,0,0,0) & \boldsymbol{v}_{23} = \frac{1}{2}(0,1,1,0,0,0) & \boldsymbol{v}_{26} = \frac{1}{2}(0,1,0,0,0,1) \\ \boldsymbol{v}_{14} = \frac{1}{2}(1,0,0,1,0,0) & \boldsymbol{v}_{34} = \frac{1}{2}(0,0,1,1,0,0) & \boldsymbol{v}_{46} = \frac{1}{2}(0,0,0,1,0,1) \\ \boldsymbol{v}_{15} = \frac{1}{2}(1,0,0,0,1,0) & \boldsymbol{v}_{35} = \frac{1}{2}(0,0,1,0,1,0) & \boldsymbol{v}_{56} = \frac{1}{2}(0,0,0,0,1,1) \end{array}$$

where $\boldsymbol{v}_{ij}$ is the vertex half-way between $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$.

The fact that we have three vertices on each side of the hyperplane, and that the edges between them are cut exactly in half, suggests that there may be a symmetry involved here, and indeed that is the case; the two halves of the 5-simplex are congruent, so P_{1a} divides the simplex exactly in half. To see this algebraically, we can give a transformation that maps the hyperplanes that form one half onto those that form the other half. The system T_0 together with H_{1a} intersects to form R_{1a}; if we introduce halfspace $-H_{1a}$ as $x_1 - x_2 + x_3 - x_4 - x_5 + x_6 \leq 0$, then T_0 together with $-H_{1a}$ intersect to form the other half of the 5-simplex. The halfspace $-H_{1a}$ can be rewritten as $-x_1 + x_2 - x_3 + x_4 + x_5 - x_6 \geq 0$; then exchanging x_1 with x_2, x_3 with x_4, and x_5 with x_6 in all the hyperplanes converts T_0 and $-H_{1a}$ into T_0 and H_{1a}, hence this also transforms one half into the other. This exchange represents the composition of three reflections; it is a rigid motion, and so is shape preserving.

To form the final region of interest, R_1, we must now slice R_{1a} by P_{1b}, the hyperplane for H_{1b}. As above, we first determine which vertices of R_{1a} are inside and which outside H_{1b}, and then compute the positions of the new vertices obtained from slicing the edges whose vertices are on opposite sides of P_{1b}. Our job is made more complicated since P_{1b} changes with β. Note, however, that the two R_1 regions obtained for β and $-\beta$ are congruent: if $-\beta$ is substituted for β in H_{1b}, and if we exchange x_1 with x_3 and x_2 with x_4, the result is the original H_{1b}, while this exchange leaves H_{1a} and the original 5-simplex unchanged. Thus, by symmetry, we need only be concerned with β in the range $0 \leq \beta \leq 1$. (Note the close connection between these arguments and the ones in theorem 5.)

Let $\boldsymbol{N}_{1b} = \langle 1-\beta, -(1+\beta), 1+\beta, -(1-\beta), -2, 2\rangle$, a normal vector for the hyperplane P_{1b} bounding the halfspace H_{1b}. Then when we intersect R_{1a} with H_{1b}, a point $\boldsymbol{x}$ of R_{1a} remains in R_1 if, and only if, $\boldsymbol{N}_{1b} \cdot \boldsymbol{x} \geq 0$. We can compute $\boldsymbol{N}_{1b} \cdot \boldsymbol{x}$ for each vertex in R_{1a} and determine the values of β (if any) for which the vertex still lies in the region. For example, $\boldsymbol{u}_1$ is a vertex of R_{1a}, and $\boldsymbol{N}_{1b} \cdot \boldsymbol{u}_1 = 1 - \beta$, so $\boldsymbol{N}_{1b} \cdot \boldsymbol{u}_1 \geq 0$ only when $1 \geq \beta$. This is true for all β in our range of interest, so $\boldsymbol{u}_1$ remains in R_1 for all β. On the other hand, $\boldsymbol{v}_{12}$ is a vertex of R_{1a}, but $\boldsymbol{N}_{1b} \cdot \boldsymbol{v}_{12} \geq 0$ only when

$0 \le \frac{1}{2}(1-\beta) - \frac{1}{2}(1+\beta) = -\beta$; i.e., when $\beta \le 0$. So $\boldsymbol{v}_{12}$ is cut off by P_{1b} for all β of interest (except $\beta = 0$). Finally, $\boldsymbol{v}_{14}$ is a vertex of R_{1a} and $\boldsymbol{N}_{1b} \cdot \boldsymbol{v}_{14} = \frac{1}{2}(1-\beta) - \frac{1}{2}(1-\beta) = 0$. This is true for all β, so $\boldsymbol{v}_{14}$ lies on the slicing hyperplane for all β (and so remains within the region R_1).

Carrying out this process for all the vertices of R_{1a}, we find that $\boldsymbol{u}_1$, $\boldsymbol{u}_3$, $\boldsymbol{u}_6$, $\boldsymbol{v}_{26}$, $\boldsymbol{v}_{34}$ and $\boldsymbol{v}_{46}$ all remain inside the region, $\boldsymbol{v}_{14}$, $\boldsymbol{v}_{23}$ and $\boldsymbol{v}_{56}$ lie on the slicing hyperplane itself, and $\boldsymbol{v}_{12}$, $\boldsymbol{v}_{15}$, $\boldsymbol{v}_{35}$ are removed by H_{1b}. This is true for all values of β between 0 and 1, so the combinatorial structure of R_1 does not change for $0 < \beta < 1$.

To find the new vertices of R_1, we find where the slicing hyperplane intersects the edges that have one vertex on each side, as we did before. In this case, however, not all possible edges are present. There are 30 edges in R_{1a}: the three original edges between $\boldsymbol{u}_1$, $\boldsymbol{u}_3$ and $\boldsymbol{u}_6$; the nine half edges from the edges having one endpoint in S^+ and one in S^-; nine edges formed by cutting the triangles having one vertex in S^+ and two in S^-; and finally, nine more from the triangles having two vertices in S^+ and one in S^-. The only ones of these that are sliced by P_{1b} are the ones between $\boldsymbol{u}_1$ and $\boldsymbol{v}_{12}$, $\boldsymbol{u}_1$ and $\boldsymbol{v}_{15}$, $\boldsymbol{u}_3$ and $\boldsymbol{v}_{35}$, $\boldsymbol{v}_{12}$ and $\boldsymbol{v}_{26}$, and $\boldsymbol{v}_{34}$ and $\boldsymbol{v}_{35}$; thus there are five new vertices.

As above, the new vertex between $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ can be computed from $\boldsymbol{p}_0 + t(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ where $t = \boldsymbol{N}_{1b} \cdot \boldsymbol{p}_0 / (\boldsymbol{N}_{1b} \cdot \boldsymbol{p}_0 - \boldsymbol{N}_{1b} \cdot \boldsymbol{p}_1)$. We obtain the following five new vertices that depend on β:

$$\boldsymbol{v}_{12}^1 = \tfrac{1}{2}(1+\beta, 1-\beta, 0, 0, 0, 0) \qquad \boldsymbol{v}_{26}^{12} = \tfrac{1}{2(1+\beta)}(1-\beta, 1+\beta, 0, 0, 0, 2\beta)$$
$$\boldsymbol{v}_{15}^1 = \tfrac{1}{3-\beta}(2, 0, 0, 0, 1-\beta, 0) \qquad \boldsymbol{v}_{35}^{34} = \tfrac{1}{2(1+\beta)}(0, 0, 1+\beta, 1-\beta, 2\beta, 0).$$
$$\boldsymbol{v}_{35}^3 = \tfrac{1}{3+\beta}(0, 0, 2, 0, 1+\beta, 0)$$

The region R_1 is the convex hull of the these vertices together with the ones from R_{1a} that are inside or on H_{1b}. Thus R_1 is the convex hull of $\boldsymbol{u}_1$, $\boldsymbol{u}_3$, $\boldsymbol{u}_6$, $\boldsymbol{v}_{14}$, $\boldsymbol{v}_{23}$, $\boldsymbol{v}_{26}$, $\boldsymbol{v}_{34}$, $\boldsymbol{v}_{46}$, $\boldsymbol{v}_{56}$, $\boldsymbol{v}_{12}^1$, $\boldsymbol{v}_{15}^1$, $\boldsymbol{v}_{35}^3$, $\boldsymbol{v}_{26}^{12}$ and $\boldsymbol{v}_{35}^{34}$.

At this point, we know the vertices of R_1, but to use the method outlined in section 6.1 to compute its volume, we need to know about its faces as well. Each face will be a 4-dimensional region lying either within one of the hyperplanes P_i, P_{1a} or P_{1b}. For each of these hyperplanes, we can determine which vertices lie in that face (they are the ones that satisfy the equation of the hyperplane), and so the face will be the convex hull of those vertices. For example, to be in the hyperplane P_1, a point's first coordinate must be zero, hence $\boldsymbol{u}_3$, $\boldsymbol{u}_6$, $\boldsymbol{v}_{23}$, $\boldsymbol{v}_{26}$, $\boldsymbol{v}_{34}$, $\boldsymbol{v}_{46}$, $\boldsymbol{v}_{56}$, $\boldsymbol{v}_{35}^3$ and $\boldsymbol{v}_{35}^{34}$ lie in this hyperplane. Thus these are the vertices of one of the 4-dimensional faces of R_1; call it F.

Since the faces of R_1 are 4-dimensional regions, they have 3-dimensional faces of their own. A similar analysis can be used to determine these faces.

For example, of the nine vertices in face F, $\boldsymbol{u}_6$, $\boldsymbol{v}_{26}$, $\boldsymbol{v}_{46}$ and $\boldsymbol{v}_{56}$ lie in P_3, so these four form one of the 3-dimensional faces of F. The 3-dimensional faces in turn have 2-dimensional faces, and so on. The complete structure of R_1 can be determined by recursively taking each n-dimensional face and calculating its $(n-1)$-dimensional sub-faces.

6.3. **Computing the Volume.** The volume-computation algorithm described in section 6.1 has us break the region R_1 into smaller regions by choosing one of its vertices and forming pyramids over the faces of R_1 that don't contain that vertex. It is to our advantage to choose a vertex that is included in the most faces so that we have the fewest pyramids to form. There are eight faces all together, and vertex $\boldsymbol{v}_{23}$ lies in six of them (the ones formed by P_1, P_4, P_5, P_6, P_{1a} and P_{1b}), so R_1 can be broken into two pyramids having $\boldsymbol{v}_{23}$ as their apex, and the faces lying in P_2 and P_3 as their bases.

To compute the volume of these two pyramids, we need to compute the volumes of the faces that form the bases of the pyramids. This entails choosing a vertex in each 4-dimensional base and breaking each into 4-dimensional pyramids. The bases of these smaller pyramids are 3-dimensional regions, which can again be broken into 3-dimensional pyramids over 2-dimensional faces. Such a breakdown for R_1 is shown in figure 10. The left-most vertex is our initial choice of $\boldsymbol{v}_{23}$, and to its right are the vertices chosen in the 4-dimensional bases of the pyramids having $\boldsymbol{v}_{23}$ as their apexes. To the right of these are the vertices chosen in the 3-dimensional bases of the 4-dimensional pyramids, and to the right of these are the 2-dimensional faces that are the bases of the resulting 3-dimensional pyramids. These are given as the vertices of polygons listed in order around each polygon.

In order to determine the volume, we start by calculating the areas of the 2-dimensional faces at the right of the diagram and work our way left. The non-triangular faces can be broken into triangles (taking the "pyramid" process one step further) and the areas of the triangles summed to get the area of the polygon. To compute the area of a triangle $\mathbf{abc}$, let $\boldsymbol{v}_1 = \mathbf{b} - \mathbf{a}$ and $\boldsymbol{v}_2 = \mathbf{c} - \mathbf{a}$ be the vectors along two sides of the triangle. Let $\boldsymbol{e}_1$ be the unit vector in the direction of $\boldsymbol{v}_1$, and let $\boldsymbol{e}_2$ be the unit vector in the direction of $\boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{e}_1)\boldsymbol{e}_1$; then $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal unit vectors that span the plane containing $\mathbf{abc}$. We can compute the area of the triangle by using $b = \boldsymbol{v}_1 \cdot \boldsymbol{e}_1$ as the base and $h = \boldsymbol{v}_2 \cdot \boldsymbol{e}_2$ as the height.

To find the volume of a pyramid over this triangle having $\mathbf{d}$ as its apex, we let $\boldsymbol{v}_3 = \mathbf{d} - \mathbf{a}$ and let $\boldsymbol{e}_3$ be the unit vector in the direction of the vector $\boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{e}_2)\boldsymbol{e}_2$. This determines an orthonormal basis for the three-space containing the pyramid. Then the height of the apex is $\boldsymbol{v}_3 \cdot \boldsymbol{e}_3$, and the volume can be computed as one third the area of the triangle times this height.

$$
\begin{aligned}
\boldsymbol{u}_1 &= (1,0,0,0,0,0)\\
\boldsymbol{u}_3 &= (0,1,0,0,0,0)\\
\boldsymbol{u}_6 &= (0,0,0,0,0,1)\\
\boldsymbol{v}_{14} &= \tfrac{1}{2}(1,0,0,1,0,0)\\
\boldsymbol{v}_{23} &= \tfrac{1}{2}(0,1,1,0,0,0)\\
\boldsymbol{v}_{26} &= \tfrac{1}{2}(0,1,0,0,0,1)\\
\boldsymbol{v}_{34} &= \tfrac{1}{2}(0,0,1,1,0,0)\\
\boldsymbol{v}_{46} &= \tfrac{1}{2}(0,0,0,1,0,1)\\
\boldsymbol{v}_{56} &= \tfrac{1}{2}(0,0,0,0,1,1)\\
\boldsymbol{v}_{12}^{1} &= \tfrac{1}{2}(1+\beta,1-\beta,0,0,0,0)\\
\boldsymbol{v}_{15}^{1} &= \tfrac{1}{3-\beta}(2,0,0,0,1-\beta,0)\\
\boldsymbol{v}_{35}^{3} &= \tfrac{1}{3+\beta}(0,0,2,0,1+\beta,0)\\
\boldsymbol{v}_{26}^{12} &= \tfrac{1}{2(1+\beta)}(1-\beta,1+\beta,0,0,0,2\beta)\\
\boldsymbol{v}_{36}^{34} &= \tfrac{1}{2(1+\beta)}(0,0,1+\beta,1-\beta,2\beta,0)
\end{aligned}
$$

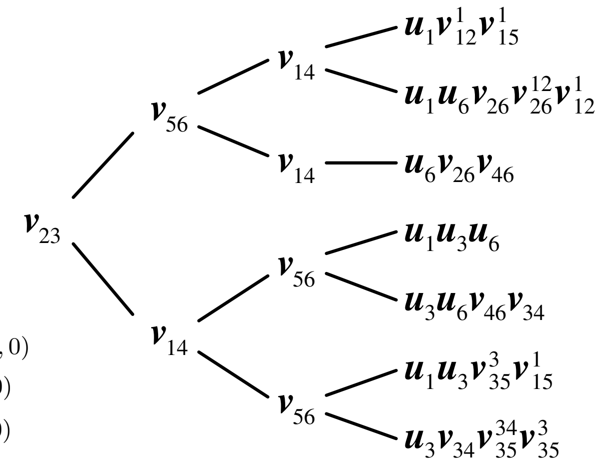

Figure 10. The pyramid structure of the region R_1 for theorem 1. The left-most vertex in the diagram is the apex of two pyramids that form the region R_1. The 4-dimensional bases of these pyramids are each divided into two pyramids with apexes given by the second column in the diagram. The third column gives the apexes of the 3-dimensional pyramids that form the bases of the 4-dimensional pyramids. The bases of these 3-dimensional pyramids are the 2-dimensional polygons listed at the right.

We continue this process by adding new basis vectors as we move left in the diagram of figure 10, adding together the results to form the total volume of the base at the next level. When the left of the diagram is reached, the result is the formula for the complete volume of the region R_1. Performing this process with the vertices we determined above for R_1 yields the formula for $\tilde{F}_1(\beta)$ for β in $[0,1]$ given in section 3. By the symmetry, the volume for $-\beta$ is also $\tilde{F}_1(\beta)$; this leads to the formula for F_1 given after theorem 1.

6.4. **The Maximum Volume.** We have computed the volume of the region as a function of the parameter β. For what value of β is this volume at a maximum? The derivative for $\tilde{F}_1(\beta)$ is

$$\tilde{F}_1'(\beta) = -\frac{\sqrt{6}\beta}{960}\cdot\frac{146+49\beta-52\beta^2-18\beta^3+2\beta^4+\beta^5}{(1+\beta)^2(3-\beta)^2(3+\beta)^2}.$$

Since $\tilde{F}(\beta)$ is valid only for β between 0 and 1, $-\sqrt{6}\beta/960$ is always negative (or zero), and the denominator of the right-hand fraction is positive. Since $\beta \geq 0$, we know $\beta^5 + 2\beta^4 + 49\beta \geq 0$, and since $\beta \leq 1$, $-\beta \geq -1$, and $-18\beta^3 \geq -18$ while $-52\beta^2 \geq -52$. Thus $146+49\beta-52\beta^2-18\beta^3+2\beta^4+\beta^5 \geq 149-52-18 = 79 > 0$. The numerator is positive, hence $\tilde{F}'(\beta) \leq 0$ for $0 \leq \beta \leq 1$. This means the volume is decreasing as β goes from 0 to 1, so the maximum volume is when $\beta = 0$. We saw before that the region for $-\beta$

is congruent to the region for β, so if we consider β in the range $-1 \leq \beta \leq 0$, the maximum again will be at $\beta = 0$. Thus the largest volume for any β in $-1 \leq \beta \leq 1$ is at $\beta = 0$.

Although this value agrees with the one produced by Van Newenhizen, her argument (as described in section 4) based on the angle between the planes is insufficient to justify this. It would require that the R_1 region for larger angles be a superset of R_1 for smaller angles. We can use the structure given in the previous section to show that this can not be the case. If we take $R_1(\beta)$ to be the region R_1 for a given β, then we will show that $R_1(1/2)$ neither contains nor is contained by $R_1(1/3)$.

To begin with, let $\boldsymbol{p}_0$ equal $\boldsymbol{v}_{15}^{1}$ for $\beta = 1/3$, that is, $\boldsymbol{p}_0 = \frac{1}{4}(3, 0, 0, 0, 1, 0)$. Note that $\boldsymbol{p}_0$ is within $R_1(1/3)$ since it is a vertex of $R_1(1/3)$. Now consider $\beta = 1/2$ and let $\boldsymbol{p}_1 = \boldsymbol{v}_{35}^{34} = \frac{1}{6}(0, 0, 3, 1, 2, 0)$. Again, $\boldsymbol{p}_1$ is in $R_1(1/2)$.

Recall that $\boldsymbol{p}_1$ is in H_{1b} if, and only if, $\boldsymbol{N}_{1b} \cdot \boldsymbol{p}_1 \geq 0$. Now when $\beta = 1/3$, $\boldsymbol{N}_{1b} = \frac{2}{3}\langle 1, -2, 2, -1, -3, 3\rangle$, so $\boldsymbol{N}_{1b} \cdot \boldsymbol{p}_1 = -1/9$. Thus $\boldsymbol{p}_1$ is not contained in H_{1b}, and therefore is not within $R_1(1/3)$. This means $\boldsymbol{p}_1 \in R_1(1/2)$ but $\boldsymbol{p}_1 \notin R_1(1/3)$, and so $R_1(1/2) \nsubseteq R_1(1/3)$, as claimed.

On the other hand, for $\beta = 1/2$, $\boldsymbol{N}_{1b} = \frac{1}{2}\langle 1, -3, 3, -1, -4, 4\rangle$, so in this case $\boldsymbol{N}_{1b} \cdot \boldsymbol{p}_0 = -1/8$. Thus $\boldsymbol{p}_0$ is not contained in H_{1b}, and therefore is not in $R_1(1/2)$. This means $R_1(1/3) \nsubseteq R_1(1/2)$, as desired.

6.5. **Adding More Planes.** The process used to compute the volume of R_1 above did not depend on the number of planes involved in forming the region. If additional planes are used, we can determine which vertices are cut off and which remain as we did above, with the new vertices being calculated as before. In order to do this, we need to know which edges exist in the region being sliced by the plane. We can determine the edges by finding the 1-dimensional "faces" of the region using the recursive process that was outlined at the end of section 6.2. Once we have the vertices that remain after all the planes have been included, we break the object into pyramids as in section 6.3, and compute the volume as before.

This method can be used to find the formulae for the volumes for the other theorems. For theorem 2, there are two new planes that slice R_1, namely

$$x_1 - x_2 - x_3 + x_4 - x_5 + x_6 > 0 \tag{H_{2a}}$$

$$\begin{aligned} 2x_1 - 2x_2 - (1-\beta)x_3 + (1+\beta)x_4 \\ - (1+\beta)x_5 + (1-\beta)x_6 > 0. \end{aligned} \tag{H_{2b}}$$

Note that there is no symmetry between $-\beta$ and β, so we must consider the entire range of values of β. The structure of the resulting region changes at $\beta = 0$ (where the slicing planes pass through vertices of the object), so two different formulae are needed, one for $\beta < 0$ and another for $\beta > 0$. The vertices together with the breakdown of the region into pyramids is given in figure 11 for $\beta < 0$ and figure 12 for $\beta > 0$. The resulting volume formulae are given as $F_2^-(\beta)$ and $F_2^+(\beta)$ in section 3.

A graph of these two volumes is shown in figure 4. Note that the shape is not symmetric about $\beta = 0$, and the maximum occurs for a β less than zero. We can locate this β by finding the zeros of the derivative of F_2. This involves finding the roots of a ninth-degree polynomial, so we can't do it algebraically. We can determine them numerically, however, to as many digits of precision as required. Doing so yields one zero in the range $-1 \leq \beta \leq 1$, namely $\beta = -0.25544192$.

For theorem 3, there are three new planes in addition to the ones that formed R_1; two fixed and one varying with β:

$$x_1 - x_2 - x_3 + x_4 - x_5 + x_6 > 0 \qquad (H_{3a_1})$$

$$x_1 - x_2 - x_3 + x_4 + x_5 - x_6 > 0 \qquad (H_{3a_2})$$

$$\begin{aligned}(1+\beta)x_1 - (1-\beta)x_2 - 2x_3 + 2x_4 \\ + (1-\beta)x_5 - (1+\beta)x_6 > 0.\end{aligned} \qquad (H_{3b})$$

In this case, there is again a symmetry between β and $-\beta$: exchanging $-\beta$ with β, x_3 with x_5 and x_4 with x_6 interchanges H_{1a} with H_{3a_2} and H_{1b} with H_{3b} while leaving H_{3a_2} unchanged. As before, this means that the volume for $-\beta$ is the same as the volume for β, so we need only consider $0 \leq \beta \leq 1$. (Again, note the close connection between the geometry here and the voting theory in theorem 5.)

The vertices and pyramid structure for the resulting volume are shown in figure 13. Carrying out the computations as before yields the volume given as $\tilde{F}_3(\beta)$ in section 3.

The derivative of this formula is negative for $0 < \beta \leq 1$, which means that the maximum value must be at $\beta = 0$. As was the case with theorem 1, since the volume for $-\beta$ is the same as for β, this means that the maximum volume for $-1 \leq \beta \leq 1$ occurs at $\beta = 0$.

For theorem 4, we have the same three fixed hyperplanes as in theorem 3, but with the following two hyperplanes that vary with β:

$$\begin{aligned}2x_1 - 2x_2 - (1-\beta)x_3 + (1+\beta)x_4 \\ - (1+\beta)x_5 + (1-\beta)x_6 > 0\end{aligned} \qquad (H_{4a})$$

$$\begin{aligned}-(1+\beta)x_1 + (1-\beta)x_2 + 2x_3 - 2x_4 \\ - (1-\beta)x_5 + (1+\beta)x_6 > 0.\end{aligned} \qquad (H_{4b})$$

As was the case with theorem 2, there is no symmetry between $-\beta$ and β, and the structure of the region changes at $\beta = 0$. The vertices together with the breakdown of the region into pyramids is given in figure 14 for $\beta < 0$

and in figure 15 for $\beta > 0$. The resulting volume formulae are given as F_4^- and F_4^+ in section 3.

A graph of these two volumes, normalized by the volume of the fixed region, is shown in figure 6. Note that the shape is not symmetric about $\beta = 0$. Although the graph appears to be decreasing, this turns out not to be the case, as mentioned in section 3. The graph actually has a minimum at approximately $\beta = .63311$, and is increasing for $\beta > .63311$, as illustrated in figure 7. As a check that no algebraic mistakes have been made, one can verify that $F_2 = 2(F_3 + F_4)$, as expected from the argument in section 5.

7. Conclusion

Several lessons may be drawn from our work here. First, when dealing with IAC, the subtleties of higher-dimensional geometry are considerable. In particular, for the case of three candidates the five- and six-dimensional geometry is sufficiently complex that we see, at this time, no simple argument that can completely replace the exact calculations in explaining why the value of β that maximizes Condorcet efficiency is less than zero. For four candidates, analogous calculations would need to be done in twenty three and twenty four dimensions, and it seems that some new idea would be needed to settle the general question for m candidates.

More broadly, our results support the Borda Rule as a strong contender for "best" scoring rule, when the goal is to have the ranking of three alternatives by scoring rule match the ranking obtained by pairwise majority rule. The assumptions of IC and IAC have some subtle differences, but as $n \to \infty$ Borda maximizes the probability of this coincidence for both assumptions.

The issue of which scoring rule is "best" at selecting the Condorcet winner, when there is one, is more problematic, however. We find that with IAC, as $n \to \infty$ the scoring rule that maximizes the conditional probability of selecting the Condorcet winner, given that there is a Condorcet winner, is *not* the Borda rule.

What are we to make of this conclusion? Some may argue that, by dethroning the mathematically more natural Borda rule in favor of a rather odd choice of scoring weights, IAC has shown itself to be unrealistic; the assumption of equally likely profiles puts too much weight on profiles that are actually very unlikely to occur. Others may never have thought of either IC or IAC as being realistic distributions in the first place, but rather cast them in the role of mathematically natural extreme cases (in which case, the more such test cases available, the better) or view them as contenders for the distribution best representing a state of *a priori* ignorance as to how people will vote.

In this connection, consider the arguments in Lepelley, Pierron and Valognes [7]. They use some observations of Berg [1] that show IC and IAC to be specific instances of a class of probability contagion models. Here, IC displays complete independence between the preferences of different voters, while IAC exhibits a small amount of dependence. Based on numerical calculations of probabilities for a number of special cases with finitely many voters, they conjecture that, in the limit, the Borda Rule only maximizes the probability of selecting the Condorcet winner when voters are assumed to have independent preferences.

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$$\begin{aligned}
\boldsymbol{v}_0 &= (1,0,0,0,0,0)\\
\boldsymbol{v}_1 &= (0,0,0,0,0,1)\\
\boldsymbol{v}_2 &= \tfrac{1}{2}(1,1,0,0,0,0)\\
\boldsymbol{v}_3 &= \tfrac{1}{2}(1,0,1,0,0,0)\\
\boldsymbol{v}_4 &= \tfrac{1}{2}(1,0,0,1,0,0)\\
\boldsymbol{v}_5 &= \tfrac{1}{2}(0,0,1,0,0,1)\\
\boldsymbol{v}_6 &= \tfrac{1}{2}(0,0,0,1,0,1)\\
\boldsymbol{v}_7 &= \tfrac{1}{2}(0,0,0,0,1,1)\\
\boldsymbol{v}_8 &= \tfrac{1}{3-\beta}(2,0,0,0,1-\beta,0)\\
\boldsymbol{v}_9 &= \tfrac{1}{3-\beta}(0,1-\beta,0,0,0,2)\\
\boldsymbol{v}_{10} &= \tfrac{1}{2(1-\beta)}(1-\beta,1+\beta,0,0,-2\beta,0)\\
\boldsymbol{v}_{11} &= \tfrac{1}{2(3+\beta)}(3+\beta,0,1+\beta,0,2,0)\\
\boldsymbol{v}_{12} &= \tfrac{1}{2(3+\beta)}(0,2,0,1+\beta,0,3+\beta)\\
\boldsymbol{v}_{13} &= \tfrac{1}{2(1-\beta)}(0,-2\beta,0,0,1+\beta,1-\beta)\\
\boldsymbol{v}_{14} &= \tfrac{1}{2(1-\beta)}(0,0,1+\beta,1-\beta,0,-2\beta)\\
\boldsymbol{v}_{15} &= \tfrac{1}{2(1-\beta)}(-2\beta,0,1-\beta,1+\beta,0,0)\\
\boldsymbol{v}_{16} &= \tfrac{1}{12}(3-\beta,3+\beta,3+\beta,3-\beta,0,0)\\
\boldsymbol{v}_{17} &= \tfrac{1}{6(1-\beta)}(-4\beta,0,(3+\beta)(1-\beta),3+\beta^2,0,0)\\
\boldsymbol{v}_{18} &= \tfrac{1}{6(1-\beta)}(0,-2\beta,1-\beta,1-\beta,2,2(1-\beta))\\
\boldsymbol{v}_{19} &= \tfrac{1}{12}(0,0,3-\beta,3+\beta,3+\beta,3-\beta)\\
\boldsymbol{v}_{20} &= \tfrac{1}{6(1-\beta)}(2(1-\beta),2,1-\beta,1-\beta,-2\beta,0)\\
\boldsymbol{v}_{21} &= \tfrac{1}{6(1-\beta)^2}(2\beta(\beta-3),0,(3-\beta)(1-\beta),3+\beta^2,2\beta(\beta-1),0)\\
\boldsymbol{v}_{22} &= \tfrac{1}{6(1-\beta)^2}(0,2\beta(\beta-1),3+\beta^3,(3-\beta)(1-\beta),0,2\beta(\beta-3))\\
\boldsymbol{v}_{23} &= \tfrac{1}{6(1-\beta)}(0,0,3+\beta^2,(3+\beta)(1-\beta),0,-4\beta)
\end{aligned}$$

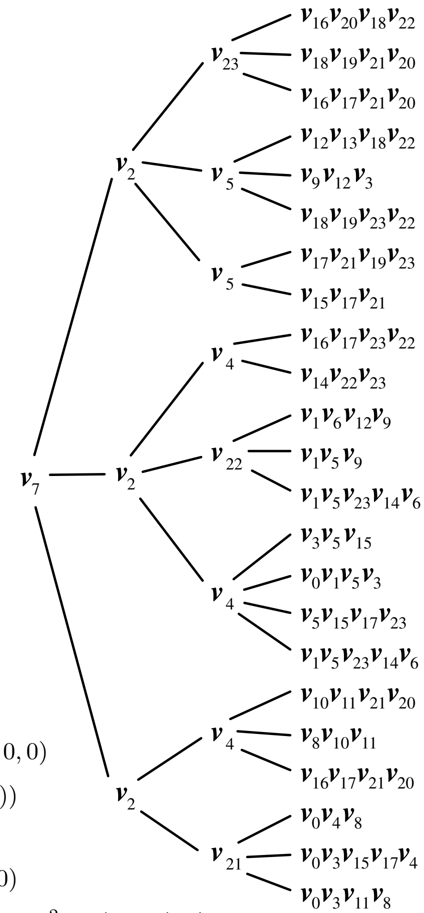

FIGURE 11. The pyramid structure for the region for theorem 2 that is formed by intersecting the 5-simplex with two additional fixed planes plus two that vary with β. This table is valid for $-1 < \beta < 0$, and can be interpreted in the same fashion as the previous figure. Note that although there is no symmetry between β and $-\beta$, the region is self-symmetric; the symmetry interchanges x_1 with x_6, x_2 with x_5 and x_3 with x_4. This exchanges the planes P_{1a} and P_{2a} and the planes P_{1b} and P_{2b}. (Note that the $\boldsymbol{v}_i$ form symmetric pairs.)

$$\begin{aligned}
\boldsymbol{v}_0 &= (1,0,0,0,0,0)\\
\boldsymbol{v}_1 &= (0,0,0,0,0,1)\\
\boldsymbol{v}_2 &= \tfrac{1}{2}(1,0,1,0,0,0)\\
\boldsymbol{v}_3 &= \tfrac{1}{2}(1,0,0,1,0,0)\\
\boldsymbol{v}_4 &= \tfrac{1}{2}(0,0,1,1,0,0)\\
\boldsymbol{v}_5 &= \tfrac{1}{2}(0,0,1,0,0,1)\\
\boldsymbol{v}_6 &= \tfrac{1}{2}(0,0,0,1,0,1)\\
\boldsymbol{v}_7 &= \tfrac{1}{2}(1+\beta,1-\beta,0,0,0,0)\\
\boldsymbol{v}_8 &= \tfrac{1}{3-\beta}(2,0,0,0,1-\beta,0)\\
\boldsymbol{v}_9 &= \tfrac{1}{3-\beta}(0,1-\beta,0,0,0,2)\\
\boldsymbol{v}_{10} &= \tfrac{1}{2}(0,0,0,0,1-\beta,1+\beta)\\
\boldsymbol{v}_{11} &= \tfrac{1}{2(1+\beta)}(1+\beta,1,\beta,0,0,0)\\
\boldsymbol{v}_{12} &= \tfrac{1}{4}(1,1,1,1,0,0)\\
\boldsymbol{v}_{13} &= \tfrac{1}{2(3+\beta)}(3+\beta,0,1+\beta,0,2,0)\\
\boldsymbol{v}_{14} &= \tfrac{1}{6(1+\beta)}(2\beta,0,0,2\beta,3+\beta,3+\beta)\\
\boldsymbol{v}_{15} &= \tfrac{1}{2(1+3\beta)}(2\beta,0,1+\beta,1+\beta,2\beta,0)\\
\boldsymbol{v}_{16} &= \tfrac{1}{2(1+3\beta)}(0,2\beta,1+\beta,1+\beta,0,2\beta)\\
\boldsymbol{v}_{17} &= \tfrac{1}{2(3+\beta)}(0,2,0,1+\beta,0,3+\beta)\\
\boldsymbol{v}_{18} &= \tfrac{1}{4}(0,0,1,1,1,1)\\
\boldsymbol{v}_{19} &= \tfrac{1}{2(1+\beta)}(0,0,0,\beta,1,1+\beta)\\
\boldsymbol{v}_{20} &= \tfrac{1}{6(1+\beta)}(4\beta,0,0,0,(1+\beta)(3-\beta),3+\beta^2)\\
\boldsymbol{v}_{21} &= \tfrac{1}{6(1+\beta)}(2\beta,0,1+\beta,1+\beta,2(1+\beta),2)\\
\boldsymbol{v}_{22} &= \tfrac{1}{6(1+\beta)}(3+\beta^2,(1+\beta)(3-\beta),0,0,0,4\beta)\\
\boldsymbol{v}_{23} &= \tfrac{1}{6(1+\beta)}(3+\beta,3+\beta,2\beta,0,0,2\beta)\\
\boldsymbol{v}_{24} &= \tfrac{1}{6(1+\beta)}(2,2(1+\beta),1+\beta,1+\beta,0,2\beta)\\
\boldsymbol{v}_{25} &= \tfrac{1}{6(1+\beta)^2}(3+\beta^2,(3+\beta)(1+\beta),0,2\beta(1+\beta),0,2\beta(3+\beta))\\
\boldsymbol{v}_{26} &= \tfrac{1}{6(1+\beta)^2}(2\beta(3+\beta),0,2\beta(1+\beta),0,(3+\beta)(1+\beta),3+\beta^2)
\end{aligned}$$

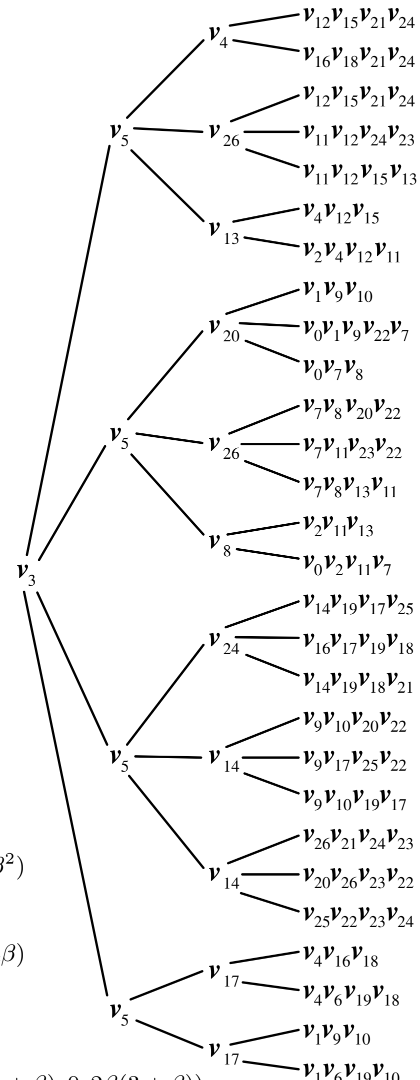

FIGURE 12. The pyramid structure for the region for theorem 2 that is formed by intersecting the 5-simplex with two additional fixed planes plus two that vary with β, this time for $0 < \beta < 1$. The structure can be interpreted in the same fashion as the previous figures. Again, for any fixed β, the region itself is symmetric; the symmetry interchanges x_1 with x_6, x_2 with x_5 and x_3 with x_4. This exchanges the planes P_{1a} and P_{2a} and the planes P_{1b} and P_{2b}. Thus the vertices also form symmetric pairs. The fact that there are more vertices in this case than when $\beta < 0$ is one way to see that there is no symmetry between the positive and negative β.

$$\boldsymbol{v}_0 = (1,0,0,0,0,0)$$
$$\boldsymbol{v}_1 = \tfrac{1}{2}(1,0,0,1,0,0)$$
$$\boldsymbol{v}_2 = \tfrac{1}{2}(1,0,0,0,0,1)$$
$$\boldsymbol{v}_3 = \tfrac{1}{2}(0,0,1,1,0,0)$$
$$\boldsymbol{v}_4 = \tfrac{1}{2}(0,0,0,1,0,1)$$
$$\boldsymbol{v}_5 = \tfrac{1}{2}(1+\beta,1-\beta,0,0,0,0)$$
$$\boldsymbol{v}_6 = \tfrac{1}{3+\beta}(2,0,1+\beta,0,0,0)$$
$$\boldsymbol{v}_7 = \tfrac{1}{3-\beta}(2,0,0,0,1-\beta,0)$$
$$\boldsymbol{v}_8 = \tfrac{1}{2(1+\beta)}(1+\beta,1,\beta,0,0,0)$$
$$\boldsymbol{v}_9 = \tfrac{1}{2(3+\beta)}(3+\beta,3-\beta,0,0,0,2\beta)$$
$$\boldsymbol{v}_{10} = \tfrac{1}{4}(1,1,1,1,0,0)$$
$$\boldsymbol{v}_{11} = \tfrac{1}{2(1+3\beta)}(1+\beta,1+\beta,0,2\beta,0,2\beta)$$
$$\boldsymbol{v}_{12} = \tfrac{1}{2(3+\beta)}(3+\beta,0,1+\beta,0,2,0)$$
$$\boldsymbol{v}_{13} = \tfrac{1}{6(1+\beta)}(2\beta,0,0,2\beta,3+\beta,3+\beta)$$
$$\boldsymbol{v}_{14} = \tfrac{1}{2(1+3\beta)}(2\beta,0,1+\beta,1+\beta,2\beta,0)$$
$$\boldsymbol{v}_{15} = \tfrac{1}{2(1+\beta)}(0,0,0,2\beta,1-\beta,1+\beta)$$
$$\boldsymbol{v}_{16} = \tfrac{1}{6(1+\beta)}(4\beta,0,0,0,(1+\beta)(3-\beta),3+\beta^2)$$
$$\boldsymbol{v}_{17} = \tfrac{1}{2(1+\beta)}(1+\beta,1-\beta,2\beta,0,0,0)$$
$$\boldsymbol{v}_{18} = \tfrac{1}{2(3-\beta)}(3-\beta,0,2,0,1-\beta,0)$$
$$\boldsymbol{v}_{19} = \tfrac{1}{6(1+\beta)}(2(1+\beta),2,2\beta,0,1+\beta,1+\beta)$$
$$\boldsymbol{v}_{20} = \tfrac{1}{12}(3+\beta,3-\beta,0,0,3-\beta,3+\beta)$$
$$\boldsymbol{v}_{21} = \tfrac{1}{6}(1,1,1,1,1,1)$$
$$\boldsymbol{v}_{22} = \tfrac{1}{6(1+\beta)}(1+\beta,1+\beta,0,2\beta,2,2(1+\beta))$$
$$\boldsymbol{v}_{23} = \tfrac{1}{6(1+\beta)^2}(2\beta(3+\beta),0,2\beta(1+\beta),0,(3+\beta)(1+\beta),3+\beta^2)$$
$$\boldsymbol{v}_{24} = \tfrac{1}{6(1+\beta)}(2\beta,0,1+\beta,1+\beta,2(1+\beta),2)$$

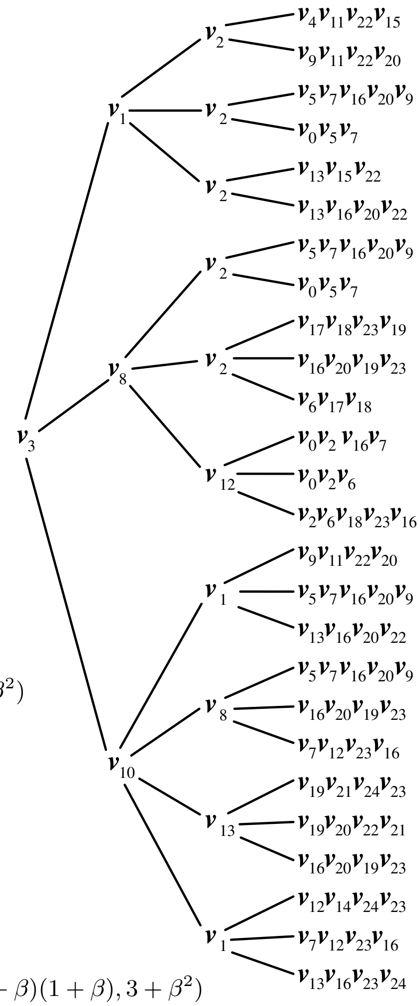

FIGURE 13. The pyramid structure for the region for theorem 3 that is formed by intersecting the 5-simplex with three additional fixed planes plus two that vary with β, for $0 < \beta < 1$. The structure for negative β is symmetric to this one, where β is replaced by $-\beta$, and where x_3 and x_5 are interchanged, as are x_4 and x_6. The pyramid structure can be interpreted in the same fashion as the previous figures.

$$\boldsymbol{v}_0 = \tfrac{1}{2}(1,1,0,0,0,0)$$
$$\boldsymbol{v}_1 = \tfrac{1}{2}(1,0,1,0,0,0)$$
$$\boldsymbol{v}_2 = \tfrac{1}{2}(1,0,0,0,0,1)$$
$$\boldsymbol{v}_3 = \tfrac{1}{2}(1-\beta,1+\beta,0,0,0,0)$$
$$\boldsymbol{v}_4 = \tfrac{1}{3+\beta}(2,0,1+\beta,0,0,0)$$
$$\boldsymbol{v}_5 = \tfrac{1}{2(3-\beta)}(3-\beta,3+\beta,0,-2\beta,0,0)$$
$$\boldsymbol{v}_6 = \tfrac{1}{2(1-\beta)}(1-\beta,1,0,0,-\beta,0)$$
$$\boldsymbol{v}_7 = \tfrac{1}{2(1-3\beta)}(1-\beta,1-\beta,0,-2\beta,0,-2\beta)$$
$$\boldsymbol{v}_8 = \tfrac{1}{4}(1,1,0,0,1,1)$$
$$\boldsymbol{v}_9 = \tfrac{1}{2(3-\beta)}(3-\beta,0,2,0,1-\beta,0)$$
$$\boldsymbol{v}_{10} = \tfrac{1}{2(1-\beta)}(-2\beta,0,1-\beta,1+\beta,0,0)$$
$$\boldsymbol{v}_{11} = \tfrac{1}{2(1-3\beta)}(-2\beta,0,-2\beta,0,1-\beta,1-\beta)$$
$$\boldsymbol{v}_{12} = \tfrac{1}{12}(3-\beta,3+\beta,3+\beta,3-\beta,0,0)$$
$$\boldsymbol{v}_{13} = \tfrac{1}{6(1-\beta)}(2(1-\beta),2,1-\beta,1-\beta,-2\beta,0)$$
$$\boldsymbol{v}_{14} = \tfrac{1}{6(1-\beta)}(1-\beta,1-\beta,2,2(1-\beta),0,-2\beta)$$
$$\boldsymbol{v}_{15} = \tfrac{1}{6}(1,1,1,1,1,1)$$
$$\boldsymbol{v}_{16} = \tfrac{1}{6(1-\beta)}(-4\beta,0,(3+\beta)(1-\beta),3+\beta^2,0,0)$$
$$\boldsymbol{v}_{17} = \tfrac{1}{6(1-\beta)^2}(2\beta(\beta-3),0,(3-\beta)(1-\beta),3+\beta^2,2\beta(\beta-1),0)$$
$$\boldsymbol{v}_{18} = \tfrac{1}{6(1-\beta)}(-2\beta,0,3-\beta,3-\beta,0,-2\beta)$$
$$\boldsymbol{v}_{19} = \tfrac{1}{6(1-\beta)}(-2\beta,0,2(1-\beta),2,1-\beta,1-\beta)$$

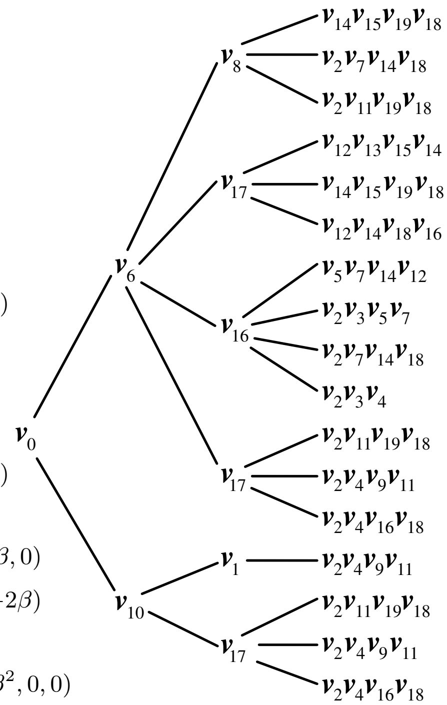

FIGURE 14. The pyramid structure for the region for theorem 4 that is formed by intersecting the 5-simplex with three additional fixed planes plus two that vary with β, for $-1 < \beta < 0$. The pyramid structure can be interpreted in the same fashion as the previous figures.

$$\boldsymbol{v}_0 = \tfrac{1}{2}(1,0,1,0,0,0)$$
$$\boldsymbol{v}_1 = \tfrac{1}{2}(1,0,0,0,0,1)$$
$$\boldsymbol{v}_2 = \tfrac{1}{2}(0,0,1,1,0,0)$$
$$\boldsymbol{v}_3 = \tfrac{1}{3+\beta}(2,0,1+\beta,0,0,0)$$
$$\boldsymbol{v}_4 = \tfrac{1}{2(1+\beta)}(1+\beta,1-\beta,2\beta,0,0,0)$$
$$\boldsymbol{v}_5 = \tfrac{1}{2(3-\beta)}(3-\beta,0,2,0,1-\beta,0)$$
$$\boldsymbol{v}_6 = \tfrac{1}{2(1+3\beta)}(2\beta,0,2\beta,0,1+\beta,1+\beta)$$
$$\boldsymbol{v}_7 = \tfrac{1}{2(3+\beta)}(2\beta,0,0,0,3-\beta,3+\beta)$$
$$\boldsymbol{v}_8 = \tfrac{1}{4}(0,0,1,1,1,1)$$
$$\boldsymbol{v}_9 = \tfrac{1}{2(1+\beta)}(0,0,0,\beta,1,1+\beta)$$
$$\boldsymbol{v}_{10} = \tfrac{1}{6(1+\beta)}(2(1+\beta),2,2\beta,0,1+\beta,1+\beta)$$
$$\boldsymbol{v}_{11} = \tfrac{1}{12}(3+\beta,3-\beta,0,0,3-\beta,3+\beta)$$
$$\boldsymbol{v}_{12} = \tfrac{1}{6}(1,1,1,1,1,1)$$
$$\boldsymbol{v}_{13} = \tfrac{1}{6(1+\beta)}(1+\beta,1+\beta,0,2\beta,2,2(1+\beta))$$
$$\boldsymbol{v}_{14} = \tfrac{1}{2(1+\beta)}(0,0,0,2\beta,1-\beta,1+\beta)$$
$$\boldsymbol{v}_{15} = \tfrac{1}{6(1+\beta)^2}(2\beta(3+\beta),0,2\beta(1+\beta),0,(3+\beta)(1+\beta),3+\beta^2)$$
$$\boldsymbol{v}_{16} = \tfrac{1}{6(1+\beta)}(4\beta,0,0,0,(3-\beta)(1+\beta),3+\beta^2)$$
$$\boldsymbol{v}_{17} = \tfrac{1}{6(1+\beta)}(2\beta,0,1+\beta,1+\beta,2(1+\beta),2)$$
$$\boldsymbol{v}_{18} = \tfrac{1}{6(1+\beta)}(2\beta,0,0,2\beta,3+\beta,3+\beta)$$

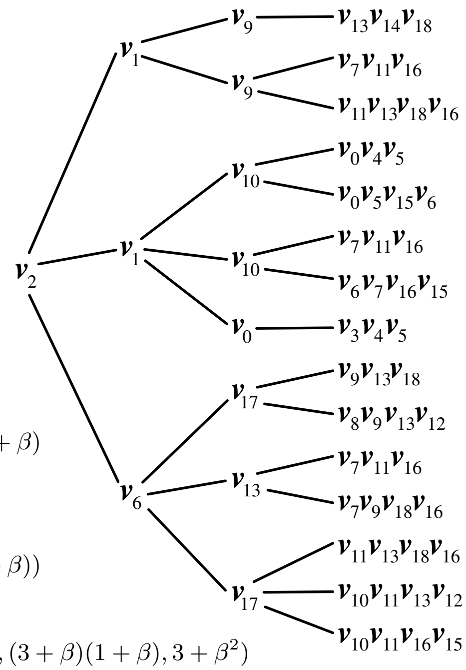

Figure 15. The pyramid structure for the region for theorem 4 that is formed by intersecting the 5-simplex with three additional fixed planes plus two that vary with β, this time for $0 < \beta < 1$. The pyramid structure can be interpreted in the same fashion as the previous figures. The fact that it is different from the structure in the previous figure indicates that there is no symmetry between the positive and negative values of β.