LEWIS CARROLL AND THE THEORY OF GAMES

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Lewis Carroll's booklet, *The Principles of Parliamentary Representation* (1st ed., Nov., 1884, 2nd ed., Jan., 1885), applies the technique of the two-person zero-sum game, which we usually associate with economic theory, to provide a theory of proportional representation and a theory of the apportionment of parliamentary seats. The entire reasoning of the booklet is expressed in quantitative terms, again on the basis of the two-person zero-sum game and the maximin criterion. A few months after the booklet had been completed Carroll made a further application of game theory, this time using the coalition game, in a very tentative way, to find the most suitable set of candidates to represent a constituency in an election.

The year 1884 was the year of the last great debate in Britain on the franchise. If we exclude Ireland, the electors belonged to one or other of the two political parties. The number of parliamentary seats was fixed, and the more seats the one party got, the fewer went to the other. This setup invites an application of the two-person zero-sum game; but to the present day, with a solitary exception,[1] Carroll has been the only writer to provide a model of this kind. Whether we say that he actually uses the two-person zero-sum game in which each of the parties acts on the maximin criterion, or on the other hand provides only a model which has the same effect as this, is a semantic question and perhaps not very important. In any case, my own answer would be that it is the two-person zero-sum game and the maximin criterion which he uses; and there can be no doubt whatever that he makes use of the coalition game.

Carroll regards the choice of an electoral system as being a problem in proportional representation. Consider, he says, the main family of electoral systems: that in which a constituency has m seats ($m \geqq 1$), and the elector is allowed v votes ($1 \leqq v \leqq m$), of which he can give no more than one to any candidate.[2] The seats will of course be awarded to the m candidates with the highest number of votes. The best or most suitable electoral system will be that member of the family which gives rise to the greatest degree of representation. Or, in his treatment, going on to quantify the problem, the best electoral system will be that member of the family which maximizes "the number of electors represented," and which consequently minimizes "the number of electors unrepresented." This requires, among other things, a definition of "the number of electors represented" (or unrepresented) in any particular election; but we will come to that later.

To get his model of the two-party system in politics, Carroll must, of course, make certain definite assumptions. He supposes that each of the two political parties knows the number of its own supporters and the number of supporters of the other party; and that each party is able to direct the voting of its supporters as it chooses, getting so many of its supporters to vote for these candidates, so many for these others, and so on, with a view to maximizing the number of seats it fills. To illustrate, let us choose as a particular example a constituency contested by two parties, A with 100 sup-

[1] Cf. Jack Sawyer and Duncan MacRae, Jr., "Game Theory and Cumulative Voting in Illinois," *Amer. Polit. Sci. Rev.*, 1962, pp. 937–40.

[2] The booklet deals also with cumulative voting, in which the elector may cumulate his v votes on one or more of the candidates.

porters and B with 70; and let us take it that this constituency has three seats ($m=3$) and the elector is allowed two votes ($v=2$). The party A may choose to put up two candidates giving each 100 votes, i.e., may use the strategy (100, 100); or it may put up three candidates and use the strategy (100, 60, 40), or the strategy (80, 70, 50), and so on.

To tackle this problem of the three-seat two-vote constituency, we might, at the present day, argue in this fashion. The set of strategies open to the party A is the set of ways of partitioning 200 votes into two parts each consisting of 100, together with the set of ways of partitioning 200 votes into three parts no one of which exceeds 100. And conceptually we may arrange this rather long series of strategies down the left-hand column of the payoff matrix. Likewise along the top row of the matrix we may arrange all the strategies open to the party B.

Now with a matrix framed in this way, suppose that A uses any given strategy open to it and B uses any given strategy. The electoral rules will then specify which of the candidates are to get the seats, and we can fill in the figure in the corresponding cell of the matrix, showing A's payoff and B's payoff. When we fill in each cell in this way, we can proceed in the usual manner by adding a column of figures to the right of the matrix to show the minimum number of seats A fills, whichever strategy it may use, and correspondingly for B. From this, if we suppose that each of the two parties acts on the maximin criterion, we can deduce the choices made by both A and B. Approaching the problem in this way we might be apprehensive that choice by each party on the maximin criterion might not give rise to an equilibrium point—and in this problem the concept of a "mixed strategy" would be meaningless. It can be shown, however, that in the electoral problem we need have no cause for apprehension and that, trivial exceptions apart, choice made on the maximin criterion must give rise to equilibrium.

This shows the logical structure of the problem and it is the way in which we might approach it today; but in Carroll's day a different approach appears to have been quite common. In 1884 almost two-thirds of the parliamentary seats were in multi-seat constituencies returning two, three, or four members, in which voting tactics were vitally important and came under supervision of the local party caucus. Already in 1853 in a pamphlet[3] eulogized by John Stuart Mill, James Garth Marshall had shown how, in what we now refer to as the m-seat v-vote constituency, each of the two parties may choose a rational strategy. Marshall had worked out a large number of arithmetical examples and, although he did not formulate these rules, it was made fairly plain that to choose a rational (maximin) strategy it was sufficient that a party should: (1) aim to fill a definite number of seats, s seats say ($v \leqq s \leqq m$); (2) put up exactly s candidates; and (3) distribute its votes among its s candidates as evenly as possible.

This provided a very direct way of choosing a rational strategy. Knowing its own strength and that of its opponent, a party needed to consider the outcome only if it put up v or $(v+1)$ or . . . or m candidates, dividing its votes among them as evenly as possible, while its opponent, behaving rationally, did the same from its side. Take, for instance, the three-seat two-vote constituency with 100 voters of whom x support the party A and $(100-x)$ support the party B. A rational strategy for A would be either

$$(x,\ x) \text{ or } \left(\frac{2x}{3},\ \frac{2x}{3},\ \frac{2x}{3}\right)$$

[3] *Majorities and Minorities: Their Relative Rights.*

—though other rational strategies may exist which have the same outcome as one of these; and a rational strategy for B would be either

$$(100 - x, 100 - x) \quad \text{or} \quad \left(\frac{200 - 2x}{3}, \frac{200 - 2x}{3}, \frac{200 - 2x}{3}\right).$$

Suppose x is known and equal to 46. A rational strategy for A would be (46, 46) or (31, 31, 30), "or" being used in the exclusive sense, and for B (54, 54) or (36, 36, 36). If A puts up two candidates with 46 votes apiece, it is bound to fill one seat against the best counterstrategy that B can bring against it, though the stategy (44, 42) say, would also secure this. Similarly a rational strategy for B to use is (54, 54), ensuring that it fills two seats against whichever strategy A may employ; and no strategy exists ensuring that B will fill more than two seats.

Marshall's own work was cast in terms of critical values—the minimum number of supporters required by a party to fill one or two or . . . or m seats, in some arithmetical example. From the above it is easy to verify that, in the three-seat two-vote constituency, to fill one seat against the best counterstrategy, a party requires the support of 40+ p.c. of the electorate, to ensure filling two seats the support of 50+ p.c. and to ensure filling three seats the support of 60+ p.c. In terms of these critical values, a party that has the support of 46 p.c. of the electorate will again be able to ensure filling one seat but not more.

Marshall's reasoning implies acceptance of what we know today as the maximin criterion in the two-person zero-sum game, and the first step in Carroll's argument is to generalize this type of reasoning for the m-seat v-vote constituency. Carroll shows algebraically the volume of support sufficient to ensure that a party which acts on the maximin criterion will fill one or two or . . . or m seats, and throughout the entire chain of reasoning of the booklet the implicit assumption is that both parties act on the maximin criterion.

The next step in the argument derives from game theory and, as we would formulate it today, makes explicit use of the maximin criterion. The purpose of the booklet is to show which constituency, i.e., which values of m and v, give the maximum degree of representation. Applying Occam's razor, Carroll avoids definition of "representation," a complex concept which he does not require, and defines only "the number of electors represented" or equally its converse, "the number of electors unrepresented," for if, out of the e electors in a constituency f are represented, $(e–f)$ will be unrepresented.

His definition of the percentage of electors unrepresented is this. Suppose that in any given election in which both parties act on the maximin criterion, a party with the support of h p.c. of the electors fills a given number of seats, but could have filled this same number of seats with the support of only h' p.c. of the electorate $(h' \leq h)$: then the votes of $(h–h')$ p.c. of the electorate who support this party are useless or "wasted" and play no part in determining the result, and $(h–h')$ p.c. of the electorate is unrepresented. Likewise, if the other party has the support of k p.c. of the electors and could have filled the same number of seats with the support of only k' p.c. $(k' \leq k)$, then a further $(k–k')$ p.c. of the electors are unrepresented. In all $(h–h')+(k–k')$ p.c. of the electorate is unrepresented.

To illustrate, take again the three-seat two-vote constituency in which one party has the support of say, 46 p.c. of the voters and, with both parties acting on the maximin criterion, fills one seat. But it could have filled one seat with the support of only 40+ p.c. of the voters. Hence 6– p.c. of the voters are unrepresented. Similarly the other party with the support of 54 p.c. of the voters fills two seats and could have done so with the support of only 50+ p.c., so that a further 4− p.c. of the voters are unrepresented. In all in the election 10− p.c. of the voters are unrepresented.

From these two premises, the argument proceeds to establish the mathematical expectation of the percentage of the electorate unrepresented in constituencies with various numbers of seats and with the elector allowed more or fewer votes. It is found[4] that in the m-seat v-vote constituency the mathematical expectation of the percentage of the electorate unrepresented diminishes as m increases and again is smaller the smaller the size of v. The commonsense conclusion to be drawn is that we want an electoral system based on large constituencies, with four or five seats, Carroll suggests, in which the elector is allowed only a single vote.

The model for the two-party system is got on the basis of various simplifications, one of them being the assumption that the elector prefers any candidate of his own party to any candidate of the other, and another that he is equally well represented whichever candidate or candidates his party chooses to put up. But before he wrote the booklet, in letters to the *St. James's Gazette*, and after he had got it completed, in a *Supplement* and *Postscript to Supplement*, Carroll employed a more general model than this. Suppose we have a multimember constituency in which, in relation to the candidates who stand, the voter's preferences are subject to no restriction, and possibly do not even follow party lines. How, with m seats available, can we choose that set of m candidates which will give rise to the fullest representation?

Carroll gives what we may term (1) a conceptual approach, (2) an operational approach, and (3) a practical scheme of election. In all three he employs the Droop quota: in this m-member constituency it is possible (in all relevant cases) for each of exactly m candidates to get a full Droop quota of votes.[5]

1. In the conceptual approach we may imagine a group of voters, each with a definite preference schedule in relation to all the candidates who stand, and each voter knowing the preference schedule of every other. We would allow the electors to form and reform themselves into coalitions, each coalition aiming to command one or two or three, etc., Droop quotas of votes, in order to fill accordingly one or two or three, etc., seats. If such coalitions were able to form and re-form by a process of contract and recontract, until a stable set of coalitions emerged from which no elector had any incentive to detach himself, we would regard the candidates returned by this set of coalitions as an optimum set of representatives for the constituency.

This would be possible in practice, however, only if (and of course here and elsewhere we are using modern terms[6] where they help to express Carroll's notions) the costs of obtaining information about the preference schedules of other voters, and about the state of the coalitions at any moment, were zero or very low, and if the costs of entering into fresh contracts with the members of any coalition were zero or were very low, and if the process could be carried out within a short period of time. In fact, except for the very small group, the costs of the elector finding out

[4] Cf. Black, "The Central Argument in Lewis Carroll's *The Principles of Parliamentary Representation*," *Papers in Non-market Decision-making*, No. 3, Fall, 1966, pp. 1–17.

[5] The Droop quota is the integer next greater than $\dfrac{\text{total number of votes cast}}{m+1}$.

[6] Cf. James M. Buchanan and Gordon Tullock, *The Calculus of Consent* (1962), Chap. 8.

the preferences of other electors, finding out the existing state of the rival coalitions, and of terminating the existing contract and entering into a new contract with another coalition would be far from zero. It is just not feasible in the ordinary election for the voters to form and re-form themselves into coalitions, one set after another, until a stable set is arrived at which will return m candidates.

2. We may therefore seek an operational approach. Take it that we have a given set of preference schedules. The voters cannot in practice enter into coalitions among themselves, but we may be able to draw up rules whereby we can discern the coalitions which the voters, acting rationally, might form. This would show which individual candidates would get the support of 1-Droop-quota coalitions to elect these individual candidates, which pairs of candidates would get the support of 2-Droop-quota coalitions to elect these pairs of candidates, and so on.

Carroll in fact sketched a few rules which would assist in picking out coalitions of this sort, but said they were incomplete; and after working longer on the problem he probably got to know that, in general, the answer to the problem is indeterminate. But, in particular examples, determination of the coalition which it would be appropriate for the voters to form seems clear enough to common sense, and Carroll gives two or three instances of this sort: in fact he employs them to show that the single transferable vote gives a different and therefore a wrong answer in these particular cases. In general, however, an operational answer to the problem is again not feasible.

3. In the practical scheme of election which he proposes for use in Great Britain, Carroll sidesteps both the previous difficulties. He argues that, in the first place, the average British elector, "Hodge, fresh from the plough!" will know which candidate he prefers to any of the others, but will be unable to rank the candidates in order of preference: his felt preferences will amount to him knowing only which of the candidates he likes most. If so, the elector should be asked to give a vote for only a single candidate. Then at the end of the election the candidates themselves, Carroll suggests, should collect up their votes and treat these votes as if they were their own private property. Any candidate with a Droop quota of votes should be elected. When this had been done the next stage would be for the candidates to meet and exchange their votes among themselves, those who had already been elected exchanging or donating their surplus votes over and above the Droop quota which had been used in their election. It would be at this stage in the process that the coalitions would be formed. The candidates would already know one another's political attitudes and, with only a few well-informed and well-practised people meeting together after an election, the transaction costs in the formation of coalitions would be very low. Thus in casting only first-preference ballots the voters would have expressed accurately their attitudes; and, at the same time, the choice of an optimum set of m candidates as envisaged by the conceptual approach, would have been attained by the actual formation of coalitions—but coalitions among the candidates and not among the voters; i.e., among people to whom coalition formation is appropriate.

We will not attempt to evaluate this suggestion of Carroll's. For our present purposes we wish only to point out that just as in the two-party contest he makes use of the two-person zero-sum game, so in the nonparty or multiparty case Carroll makes use of the basic notions of what we know today as the coalition game with ordinal utilities.